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Algebraic topology: On results of quotient for topological modules

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ABSTRACT

Our main interest in this work is to study on topological group and topological groupoid spaces. We give new some results of certain types for topological group which are source proper group space denoted by (SPHΓ-Space), for topological groupoid are source proper topological groupoid denoted by (SPHG-Space) and for H-Space (H,π, D) are source proper groupoid space, denoted by (SPHH-Space). The important point is to get relationship of SPHΓ, SPHG-Space and SPHH-Space.

Keywords: Group space, proper group space, topological group, topological groupoid

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1. Introduction

In this paper, we study SPHΓ-Space, SPHG-Space and SPHH-Space and they properties for this purpose, we divide this work into section: In section one, we give the definition topological group, topological groupoid and study the properties of these spaces. In section two, we provided with several proposition about the relationship of SPHΓ-Space, SPHG-Space and SPHH-Space respectively. Finally, propositions in this project are presented and debated.

2. Topological group space

In this section, we give primary concepts of this research, Let (H,D) be a topological groupoid, M be a topological Space, $\pi: M \to D$ be a continuous map and Let $H * M$ denote the fiber product of α and π over D. A left action of H on (M, π, D) is a continuous map $\theta^*: H * M \to M$ such that (i) $\pi(\theta^*(h,z)) = \beta(h)$ for each $(h,z) \in H * M$. (ii) $\theta^*(w(\pi(z)),z) = z$ for each $z \in M$.(iii) $\theta^*\left(h(\theta^*(h,z))\right) = \theta^*(\gamma(h,\acute{h}),z)$ for each $(h,\acute{h}) \in H * H$ and $(h,z) \in H * M$. the bundle (M,π,D) together with action θ^* is called groupoid space and is denoted by H-space. [5], [6]. Let (M,π,D) be a H-space then: The subset $H(Z) = \{\theta^*(h,z) | h \in H_{\pi(z)}\}$ is the orbit of z under H. The action of H on (M,π,D) is free if for each$(h,z) \in H * M$ the relation $\theta^*(h,z) = z$ implies his unity. The action of H on (M,π,D) is a transitive if for each $,\acute{z}) \in M \times M$, there is $h \in H$ such that $\acute{z} = \theta^*(h,z)$ [3]. Let (M,π,D) be a H-space then: We say that (M,π,D) is free H-space if the action of H on (M,π,D) is free. We say that (M,π,D) is transitive H-space if the action of H on (M,π,D) is transitive. [2]. A topological groupoid (H,D) is called source proper groupoid (SPHG-Space) if: The source map $\alpha: H \to D$ is a proper. The base space D is a Hansdorff. [4] , [3]. A $\Gamma -$ Space M is called source proper group space (SPH Γ-Space) if: The action groupoid $(M \times \Gamma, \mathrm{M})$ is SPH-Space. M is free $\Gamma -$ Space. [7], [1]. Let (H,D) be an SPH-Space then the $\alpha -$ fiber space H_x is SPH_x H_x $-$space for every $x \in D$. A H-space (M,π,D) is called source proper topological groupoid space(SPHH-Space) if: The action groupoid $(H * \mathrm{M}, \mathrm{M})$ is SPH-Space. (M,π,D) is free and transitive H-Space.

3. The main results of SPHΓ-Space, SPHG-Space and SPHH-Space

In this part, we give several properties about the relationship of SPHΓ-Space, SPHG-Space and SPHH-Space.

Proposition (1):

Let (M,π,D) be H-Space and $(g \circ f, g_\circ \circ f_\circ): (\acute{\acute{H}}, \acute{D}) \to (H, D)$ be a morphism of groupoid then $(M \times \acute{\acute{D}}, \acute{\pi} = P_2, \acute{\acute{D}})$ is $\acute{H}$ −Space where $M \times \acute{\acute{D}}$ is the fiber product of π and $g_\circ \circ f_\circ$ $over\ D, P_2: M \times_D \acute{\acute{D}} \to \acute{\acute{D}}$.

Proof :

Let $\acute{\acute{H}} * \left(M \times \acute{\acute{D}}\right)$ denoted the fiber product of $\acute{\acute{\alpha}}$ and $\acute{\acute{\pi}} = P_2$ and define: $\psi: \acute{\acute{H}} * M \times_D \acute{\acute{D}} \to M \times_D \acute{\acute{D}}$ by $\psi(\acute{\acute{h}}, \left(z, \acute{\acute{\alpha}}\left(\acute{\acute{h}}\right)\right) = \theta^*\left(g \circ f\left(\acute{\acute{h}}\right), Z\right), \acute{\acute{\beta}}\left(\acute{\acute{h}}\right)) \in$ $M_D \times \acute{\acute{D}}$ since $\pi(\theta^*\left(g \circ f\left(\acute{\acute{h}}\right), Z\right) = \left(g \circ f(\acute{\acute{h}})\right) = (g \circ f)_\circ(\acute{\acute{\beta}}\left(\acute{\acute{h}}\right))$ and θ^* is a low of action of H on (M,π,D), ψ is continuous action of $\acute{\acute{H}}$ on $(M_D \times \acute{\acute{D}}, \acute{\acute{\pi}}, \acute{\acute{D}})$ since: $\acute{\acute{\pi}}(\psi(\acute{\acute{h}}, \left(z, \acute{\acute{\alpha}}(\acute{\acute{g}})\right) = P_2(\theta^*\left(g \circ f\left(\acute{\acute{h}}\right), Z\right), (\acute{\acute{\beta}}\left(\acute{\acute{h}}\right)) = \acute{\acute{\beta}}\left(\acute{\acute{h}}\right)$, $\psi(\acute{\acute{w}}\left(\acute{\acute{\pi}}(z, \acute{\acute{b}})\right)), \left(z, \acute{\acute{b}}\right) = (\theta^*\left(g \circ f)\acute{\acute{w}}\left(\acute{\acute{b}}\right), Z\right), \acute{\acute{\beta}}\left(\acute{\acute{w}}\left(\acute{\acute{b}}\right)\right) = \theta^*\left(g_\circ \circ f_\circ)\left(\acute{\acute{b}}\right), Z\right)$, $\acute{\acute{\beta}}\left(\acute{\acute{w}}\left(\acute{\acute{b}}\right)\right) = (z, \acute{\acute{b}})$ $\psi(\acute{\acute{h}}, \psi(\acute{\acute{h}}, \left(z, \acute{\acute{\alpha}}\left(\acute{\acute{g}}\right)\right) = \psi(\acute{\acute{h}}, \theta^*\left(g \circ f\left(\acute{\acute{h}}\right), Z\right)$, $\acute{\acute{\beta}}\left(\acute{\acute{h}}\right)$ = $\psi(\acute{\acute{h}}, \theta^*\left(g \circ f\left(\acute{\acute{h}}\right), Z\right)$, $\acute{\acute{\alpha}}\left(\acute{\acute{h}}\right))$ $=\theta^*\left(g \circ f\left(\acute{\acute{h}}\right)\right)$, $\theta^*(g \circ f)\left(\acute{\acute{h}}\right), Z)), \acute{\acute{\beta}}\left(\acute{\acute{h}}\right)$ $=\left(\theta^*(\gamma(g \circ f\left(\acute{\acute{h}}\right), g \circ f\left(\acute{\acute{h}}\right)\right), Z, \acute{\acute{\beta}}\left(\acute{\acute{h}}\right) = \left(\theta^*(g \circ f(\acute{\gamma}\left(\acute{\acute{h}}, \acute{\acute{h}}\right)), Z\right), \acute{\acute{\beta}}\left(\acute{\acute{h}}\right) = \left(\theta^*(g \circ (\gamma\left(\acute{\acute{h}}, \acute{\acute{h}}\right)), Z\right), \acute{\acute{\beta}}\left(\acute{\gamma}\left(\acute{\acute{h}}, \acute{\acute{h}}\right)\right)) = \psi(\acute{\gamma}\left(\acute{\acute{h}}, \acute{\acute{h}}\right), (Z, \alpha\left(\acute{\gamma}\left(\acute{\acute{h}}, \acute{\acute{h}}\right)\right)$.

Consider the following diagram in Figure 1:

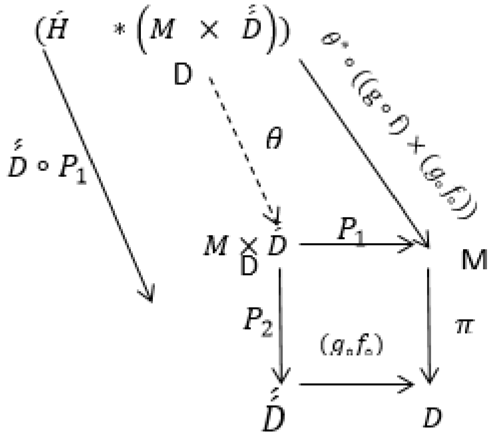

Figure 1. Diagram of Proposition (1)

In which $\pi \circ \theta^* \circ (g \circ f) \times P_2)\ (\acute{\acute{h}}, \left(z, \acute{\acute{\alpha}}\left(\acute{\acute{h}}\right)\right) = (g_\circ \circ f_\circ) \circ \acute{\acute{D}} \circ P_1(\acute{\acute{h}}, (z, \acute{\acute{\alpha}}\left(\acute{\acute{h}}\right))$ since $\pi\left(\theta^*(g \circ f)\left(\acute{\acute{h}}\right), z\right) = \acute{\beta}\left(g \circ f\left(\acute{\acute{h}}\right)\right) = (g_\circ \circ f_\circ)(\acute{\acute{\beta}}\left(\acute{\acute{h}}\right))$. $\left((M, \pi, D) is\ H - Space\right)$. Hence, there exists a unique morphism $\theta = \psi: \acute{\acute{H}} * (M \times_D \acute{\acute{D}}) \to (M \times_D \acute{\acute{D}})$ given by: $\theta^*\left(\acute{\acute{h}}, \left(z, \acute{\acute{\alpha}}\left(\acute{\acute{h}}\right)\right) = \theta^*(g \circ f)\left(\acute{\acute{h}}\right), z\right), \acute{\acute{\beta}}\left(\acute{\acute{h}}\right))$ making the whole diagram commutative in M by the universal property of fiber product.

Proposition (2) Let M be an SPH $\Gamma -$ Space then $(M, \pi, M/\Gamma)$ is SPHG $((M \times M/\Gamma) * \mathrm{M}, \mathrm{M})$ [where$((M \times M/\Gamma) * \mathrm{M}, \mathrm{M})$] is action groupoid.

Proof: Let $H = M \times M/\Gamma$ and $\mathrm{H} * \mathrm{M}$ denote the fiber product of α and $\pi = M \to M/\Gamma$ over M/Γ which is the subset of $H \times M$ of element $\left([(\acute{z}, z)], \acute{\acute{z}}\right)$, with $\acute{\acute{z}} = \theta(z, r)$, where θ is a law of action of Γ on M. Define $\psi: H * M \to M$ by: $\psi(([\acute{z}, z)], \acute{\acute{z}})) = \theta(\acute{z}, \delta^*(z, \acute{\acute{z}}))$, where δ^* is the map $\delta^*: (M \times_D \acute{\acute{D}}) \to \Gamma$ is continuous where M be SPH $\Gamma -$ Space. ψ is free and transitive continuous action of H on $(M, \pi, M/\Gamma)$

since: $\pi(\psi([(\acute{z},z)],\acute{\acute{z}})) = \pi(\theta\left(\acute{z},\delta^*(z,\acute{\acute{z}})\right) = \pi(\acute{z}) = \beta([(\acute{z},\acute{z})]).\psi\left(w(\pi(Z)),Z\right) = \psi([(z,z)],z) = \theta\left(z,\delta^*(z,z)\right) = \theta(z,e) = z.$

(i) $\psi[(\acute{z},z)],\psi([\acute{\acute{z}},\acute{\acute{z}}] = \psi([(\acute{z},z)],\theta\left(\acute{\acute{z}},\delta^*(\acute{\acute{z}},z_1)\right),\theta(\acute{z},\delta^*(z,\theta\left(\acute{\acute{z}},\delta^*(\acute{\acute{z}},z_1)\right)$
$= \theta(\acute{z},\delta^*\left(z,\acute{\acute{\acute{z}}}\right)) = \psi(\gamma[(\acute{z},z)],[\left(\acute{\acute{z}},\acute{\acute{\acute{z}}}\right)])),z_1)$

Since $\pi(z) = \pi(\acute{\acute{z}}).\Psi$ is a free action since if $\psi([(\acute{z},z)],z) = Z$ then $\theta\left(\acute{z},\delta^*(\acute{\acute{z}},z)\right) = Z$ and $\theta(\acute{\acute{z}},\gamma) = Z$ and $\theta(\acute{z},\gamma) = Z \Longrightarrow \theta(\acute{\acute{z}},\gamma) = \theta(\acute{z},\gamma) \Longrightarrow \acute{z} = \acute{\acute{z}}$ and then $[(\acute{z},\acute{z})]$ is unity.Ψ is transitive action since if $(\acute{z},z) \in M \times M$ then $\psi([(\acute{z},z)] = \theta\left(\acute{z},\delta^*(z,z)\right) = \theta(\acute{z},e) = \acute{z}$. Now to show that $(H * M, M)$ is SPH-groupiod: The base space M is Hausdorff. $H * M = (\alpha \times \pi)^{-1}(\Delta\, M/\Gamma)$. The source map α is proper since $(M \times M/\Gamma)\,,M/\Gamma)$ is SPH- groupiod, since M/Γ is compact. Hence $\Delta\, M/\Gamma$ is compact in $M/\Gamma \times M/\Gamma$ (closed subspace of compact space) and the map $\pi: M \to M/\Gamma$ is proper. Therefore $H * M$ is compact since $\alpha \times \pi$ is proper map and then it's source map is proper.

Proposition (3): Let $M \times M/\Gamma, M/\Gamma$ be on SPH Γ – space then M is $SPHH_{\pi(Z)}\ H_{\pi(Z)}$ −Space , for all $z \in M$. Where $\pi: M \times M/\Gamma \to M/\Gamma$.

Proof: The map $\eta_Z: H_{\pi(Z)} \to M, \eta_Z(h) = \theta^*(h,Z)$ where θ^* is the law of action of H-space and η_Z is closed map since for every closed subset A of $H_{\pi(Z)}$ then A is compact ($G_{\pi(Z)}$ is compact) and then $\eta_Z(A)$ is compact and consequently $\eta_Z(A)$ is closed (M is Hausdorff since $(H * M, M)$ is SPHH-space). Hence η_Z is homeomorphism, now define , $\phi_z = MX_{\pi(Z)}\,H_{\pi(Z)} \xrightarrow{\eta_Z^{-1}\times I} H_{\pi(Z)}X_{\pi(Z)}H_{\pi(Z)} \xrightarrow{\acute{\gamma}} H_{\pi(Z)} \xrightarrow{\eta_Z} M,$ by:$\phi_z(\acute{z},r) = \eta_Z(\gamma(h,r))$ where h is a unique element such that $h = \eta_Z^{-1}(\acute{z})$ and $\acute{\gamma} = \gamma|_{H_{\pi(Z)}X_{\pi(Z)}H_{\pi(Z)}}$ M is $free_{\pi(Z)}H_{\pi(Z)}$ −Space by ϕ_z since: $\phi_z(\acute{z},w\left(\pi(Z)\right) = \phi_z(\eta_Z(h),w\left(\pi(Z)\right)) = \eta_Z(\gamma(h,w\left(\pi(Z)\right) = \eta_Z(\gamma(h,w\left(\alpha(h)\right)) = \eta_Z(h) = \acute{z}$ where $w(\pi(Z))$ is unity $in_{\pi(Z)}\ H_{\pi(Z)}$
$\phi_z(\acute{z},\gamma(r_1,r_2)) = \phi_z(\eta_Z(h),\gamma(r_1,r_2)) = \eta_Z(\gamma(h,\gamma(r_1,r_2))$
$= \eta_Z(\gamma(h,r_1),r_2) = \phi_z(\phi_z(\eta_Z(h),r_1),r_2) = \phi_z(\phi_z(\acute{z},r_1),r_2).$ If $\phi_z(\acute{z},r) = \acute{z}$ then $\phi_z(\eta_Z(h),r) = \eta_Z(h)$ ((η_Z injective)) and then γ is unityϕ_z is continuous since it is the composition of continuous map. Now to show the action groupoid $(M \times_{\pi(Z)}\ H_{\pi(Z)}\,, M)$ is SPHG-Space: M is Housdorff space since $(M,\pi\,,M/\Gamma)$ is SPHG-Space. (a)The Source map $\alpha_Z: M \times_{\pi(Z)}\ H_{\pi(Z)} \to M, \alpha_Z(\acute{z},r) = \acute{z}$ is a proper map by using the following commutative diagram:

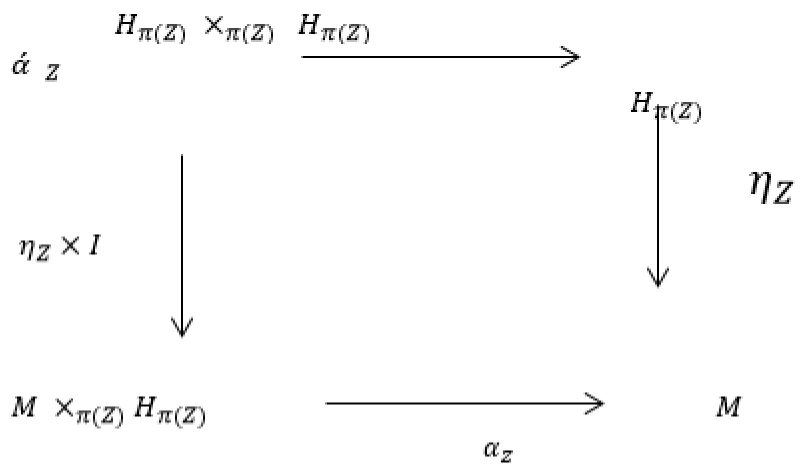

Figure 2. Diagram of Proposition (3)

Where $\acute{\alpha}_Z$ is a Source map of the action groupoid $(H_{\pi(Z)} \times_{\pi(Z)}\ H_{\pi(Z)})$ which is proper map and then we have α_Z is proper map.

Proposition (4):

Let M be an SPHG-Space then
$(M \times M /_{\pi(Z)} H_{\pi(Z)} \ , M /_{\pi(Z)} H_{\pi(Z)})$ is SPH-Space for every $z \in M$.

Proof:

M is $SPH_{\pi(Z)}\, H_{\pi(Z)}$ – Space for every $z \in M$ we have Ehressman groupoid $(M \times M /_{\pi(Z)} H_{\pi(Z)} \ , M /_{\pi(Z)} H_{\pi(Z)})$ is SPHG-Space for every $\in M$.

Proposition (5):

Let (H, D) be an SPH- groupoid and $(M \times M /_{\pi(Z)} H_{\pi(Z)})$ be SPHG-Space then the map $\pi: M \to D$ is proper map.

Proof:

Consider the following commutative diagram in M as in Figure 3.

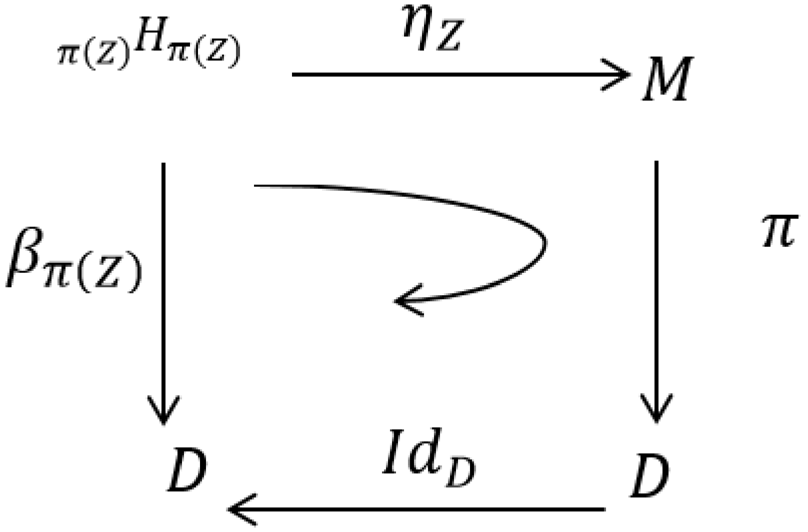

Figure 3. Diagram of Proposition (4)

In which η_Z is an homeorphism given proposition (4.1.10) and $\beta_{\pi(Z)} = w|_{\pi(Z)H_{\pi(Z)}}$ is proper map since (H,D) is SPHG-Space then $\pi \circ \eta_Z$ is proper map and then π is proper map.

4. Conclusion

New findings from certain topology groups in this paper were given, the proper grouping space denoted by (SPHΓ-Space), the proper topological groupoid source is denoted by (SPHG space), and H-space (H,π, D), the correct groupoid source denoted by groupoid source, indicates the correct topological groupoids (SPHH-Space). The point is to have SPHΓ, SPHG-Space and SPHH-Space associations that are mathematically important.

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