The random coefficient autoregressive model with seasonal volatility innovations (RCA-SGARCH)

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ABSTRACT

This paper dealt with the autoregressive model when the coefficient is random. The residuals series of the model exhibit two behaviors, kurtosis, and volatility. These volatilities are usually seasonal in the real financial data, which always uses GARCH models. So, the use of RCA and GARCH models together will provide an appropriate framework to study and analysis of time-varying volatility as well as the presence of seasonal effects in financial series. Applying copper’s daily economic close prices when the errors series are distributed, as usual, $t(3)$ and $t(7)$ distributions are achieved. Therefore, the RCA(1) model, when residuals follow the GARCH(1, 0)x(0, 1)$_5$ model together, is the appropriate model.

Keywords: RCA(1), GARCH, volatility, kurtosis.

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1. Introduction

The volatility can generally be defined as deviations and unexpected movements that appear in time series, especially financial ones. Therefore, most financial economists are concerned with return volatilities as they measure risk.

Other studies have shown that the pattern of much financial time series, such as stock returns, exhibit two behaviors, leptokurtosis, and time-varying volatility (Engle, [8]; Nicholls & Quinn, [13]; Bollerslev, [3]). Thus, the appropriate model for formulating the behaviors together is the generalized autoregressive conditional heteroscedasticity (GARCH) and the random coefficient autoregressive (RCA) models. Thavaneswaran et al. [15] derive the kurtosis of various GARCH models such as non-Gaussian GARCH, nonstationary, and random coefficient GARCH. Thavaneswaran et al. [14] derive the general properties for RCA models with GARCH innovations such as mean, variance, and kurtosis under autoregressive assumptions. Frank et al. [10] derive the kurtosis of RCA with seasonal GARCH, the variance of the $l$-steps ahead forecast errors, and the kurtosis of the error distribution. Gorka [6] found out that the RCA-GARCH models can be successfully used for pricing options for the dynamics of the volatility. Goryainov and Goryainova [11] proved the asymptotic normality of the minor absolute deviations estimate for the autoregressive models with a random coefficient. In much financial time series, such as foreign exchange rates, volatility or seasonal volatility effects and conditional non-normality can induce the leptokurtosis typically observed in economic data. These features are not appropriate for well-known models when used in future forecasting. So, other proper models must be used to study volatility or its seasonal effects in financial markets. Accordingly, RCA models have been used with the seasonal GARCH model as they provide this framework.

Our research is designed to study and analyze the Random coefficient Autoregressive Model with seasonal volatility innovations (RCA-SGARCH) and using real data of daily world copper price series (minimum prices) to estimate the appropriate studied model and then compare it with the same process when the errors series distributed as $T$- distribution with degree of freedom (3) and (5).
2. **RCA models**

The Random coefficient autoregressive time series were introduced by Nicholls and Quinn [13], and some of their properties have been studied recently by Appadoo et al. [2]. RCA models exhibiting extended memory properties have been considered in Leipus and Surgailis [12]. A sequence of random variables \{y_t\} is called an RCA(1) time series if it satisfies the equations:

\[ y_t = (\beta + b_t) y_{t-1} + e_t \]  

(1)

\( t \in \mathbb{Z}, \text{ where } \mathbb{Z} \text{ denotes the set of integers, } \beta \text{ is a real parameter and and} \)

\[ (i) \quad \begin{pmatrix} b_t \\ e_t \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_b^2 & 0 \\ 0 & \sigma_e^2 \end{pmatrix} \right) \]

\[ (ii) \quad \beta^2 + \sigma_b^2 < 1. \]

Where \{b_t\} and \{e_t\} are the errors sequences in the model. According to Nicholls and Quinn [13], condition (ii) is necessary and sufficient for the second-order stationarity of \{y_t\}.

The RCA model parameters are usually estimated using the least squares method [1,4].

**Theorem 1.** Let \{y_t\} be an RCA(1) time series satisfying conditions (i) and (ii), and let \( y \) be its covariance function. Then,

(a) \( E y_t = 0, \quad E y_t^2 = \frac{\sigma_y^2}{1-\beta^2-\sigma_b^2} \), the kth lag autocovariance for \( y_t \) is given by \( y_y(k) = \frac{\beta^k \sigma_b^2}{1-\beta^2-\sigma_b^2} \) and the autocorrelation for \( y_t \) is \( \rho_k = \beta^k \) for all \( k \in \mathbb{Z} \).

(b) If \{b_t\} and \{e_t\} are normally distributed random variables and if \( e_t \) and \( b_t \) are correlated with correlation coefficient \( \rho \), then the kurtosis \( K^{(y)} \) of the RCA process \{y_t\} is given by

\[
K^{(y)} = \frac{6 \left( \sigma_b^2 + \beta^2 \right) \left[ 1 - \beta^3 - 3\beta \sigma_b^2 \right] + 72 \beta^3 \rho^2 \sigma_b^2 + 3 \left[ 1 - (\beta^2 + \sigma_b^2) \right] \left[ 1 - \beta^3 - 3\beta \sigma_b^2 \right]}{\left[ 1 - \beta^3 - 3\beta \sigma_b^2 \right] \left[ 1 - 6\beta^2 \sigma_b^2 - \beta^4 - 3\sigma_b^4 \right]} \\
\end{aligned}
\]

\[
^* [1 - (\beta^2 + \sigma_b^2)] .
\]  

(2)

On the other hand, if \( e_t \) and \( b_t \) are uncorrelated, then \( \rho = 0 \), and the kurtosis \( K^{(y)} \) of the RCA process \{y_t\} is given by:

\[
K^{(y)} = \frac{6 [1 - (\beta^2 + \sigma_b^2)]}{[1 - 6\beta^2 \sigma_b^2 - \beta^4 - 3\sigma_b^4]} 
\]  

(3)

And if \( e_t \) and \( b_t \) are correlated with correlation coefficient \( \rho \), then the skewness \( S^{(y)} \) of the RCA process \{y_t\} is given by:

\[
S^{(y)} = \frac{6 \beta \rho \sigma_b \left[ 1 - (\beta^2 + \sigma_b^2) \right]^{1/2}}{\left[ 1 - \beta^3 - 3\beta \sigma_b^2 \right]} 
\]  

(4)

If \( e_t \) and \( b_t \) are uncorrelated, then \( \rho = 0 \), and the skewness \( S^{(y)} = 0 \).

3. **GARCH model**

The general class of these models which introduced by Bollerslev (1986) [3] can be written as:

\[
e_t = \sqrt{h_t} Z_t \\
h_t = \omega + \sum_{i=1}^{p} \alpha_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j},
\]  

(5)

(6)
Where $h_t$ is the conditional variance of the returns, $Z_t$ is (i.i.d.) random variables with $EZ_t = 0$ and $Var(Z_t) = 0$. Let $u_t = e_t^2 - h_t$ and $\sigma_t^2$ is the variance of $u_t$. Then the model could be as:

$$e_t^2 - u_t = \omega + \sum_{i=1}^{p} \alpha_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_j (e_{t-j}^2 - u_{t-j})$$

$$[1 - \sum_{i=1}^{p} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j] e_t^2 = \omega - \sum_{j=1}^{q} \beta_j u_{t-j}$$

$$\phi(B)e_t^2 = \omega + \beta(B)u_t,$$  \hspace{1cm} (7)

Where $\phi(B) = \sum_{i=1}^{r} \phi_i B^i$, $\phi_i = (\alpha_i + \beta_i)$, $\beta(B) = \sum_{j=1}^{Q} \beta_j$.

4. **Multiplicative seasonal GARCH models**

This model can be written as follows [7]:

$$e_t = \sqrt{h_t} Z_t \hspace{1cm} \text{(8)}$$

$$\theta(B)\theta(L)h_t = \omega + \alpha(B)e_t^2 \hspace{1cm} \text{(9)}$$

Where $\{Z_t\}$ is (i.i.d.) random variables with $EZ_t = 0$ and $Var(Z_t) = 0$.

$$\alpha(B) = \theta(B)\theta(L) - \phi(B)\Phi(L) \hspace{1cm} \text{(10)}$$

$$\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i \hspace{1cm} \theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i \hspace{1cm} \Phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i \hspace{1cm} \theta(L) = 1 - \sum_{i=1}^{q} \theta_i L^i \hspace{1cm} \text{and } L = B^s \hspace{1cm}$$

Letting $u_t = e_t^2 - h_t$, then equation (9) may be written as a seasonal ARMA $(p, q) \chi (P, Q)$ of $e_t^2$.

$$\theta(B)\theta(L)[e_t^2 - u_t] = \omega + [\theta(B)\theta(L) - \phi(B)\Phi(L)]e_t^2$$

Then,

$$\phi(B)\Phi(L) e_t^2 = \omega + \theta(B)\theta(L)u_t \hspace{1cm} \text{(10)}$$

5. **RCA - SGARCH Model**

The RCA - SGARCH model has been suggested by (Frank, Grahramani & Thavaneswarant, 2011 [11]). Then, by using the same transformation when obtaining equation (10), the general form of the model is given by:

$$y_t = (\beta + b_t) y_{t-1} + e_t \hspace{1cm} \text{(11)}$$

$$e_t = \sqrt{h_t} Z_t \hspace{1cm} \text{(12)}$$

$$\phi(B)\Phi(L) e_t^2 = \omega + \theta(B)\theta(L)u_t \hspace{1cm} \text{(13)}$$

Where $Z_t$, $\phi(B)$, $\Phi(L)$, $\theta(B)$, and $\theta(L)$ were defined in section 6, and $e_t^2$, as given in (10), is stationary.

Consider the RCA(1) — SGARCH $(1, 0) \chi (0, 1)$ process:
\[ y_t = (\beta + b_t) y_{t-1} + e_t \]  
(14)

\[ e_t = \sqrt{h_t} Z_t \]  
(15)

\[ (1 - \phi B) e_t = \omega + (1 - \Theta L) u_t \]  
(16)

where \( u_t = e_t^2 - h_t \), and \( \psi \)-weights are given by:

\[ \psi_1 = \phi \ldots \psi_{s-1} = \phi^{s-1}, \psi_s = (\phi^s - \theta), \psi_{s+j} = \phi^j \psi_s, j \geq 1. \]

It can be shown that

\[ \sum_{j=0}^{\infty} \psi_j^2 = \frac{[1+(\phi^s-\theta)^2]}{1-\phi^s}. \]

Then we have the following expectations of to find the kurtosis of the process.

\[ E(y_t^2) = \frac{E(e_t^2)}{[1 - \beta^2 - \sigma_b^2]} \]

\[ E(y_t^4) = \frac{6 \left[ \sigma_b^2 + \beta^2 \right] (E(e_t^2))^2}{[1 - (\beta^2 + \sigma_b^2)][1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} + \frac{E(e_t^4)}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} \]

\[ k(y) = \frac{6 \left[ \sigma_b^2 + \beta^2 \right] [1 - \beta^2 + \sigma_b^2]}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} + \frac{3[1 - (\beta^2 + \sigma_b^2)]}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} \]

where,

\[ \frac{E(h_t^2)}{\{E(h_t)\}^2} = \frac{1}{E(Z_t^4) - (E(Z_t^4) - 1) \sum_{j=0}^{\infty} \psi_j^2} \]  
(18)

For a standard normal variate \( E(Z_t^4) = 3 \), then the kurtosis of \( y_t \) is:

\[ k(y) = \frac{6 \left[ \sigma_b^2 + \beta^2 \right] [1 - (\beta^2 + \sigma_b^2)]}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} + \frac{3[1 - (\beta^2 + \sigma_b^2)]}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} \]  
(19)

The forecast error variance of the series \( y_{n+1} \) is denoted by \( e_n^{(y)}(l) \), where \( (l) \) is the steps-ahead, and the variance of \( e_n^{(y)}(l) \) for the \( RCA(1) - GARCH(1,0)x(0,1) \) model is given by:

\[ var \left[ e_n^{(y)}(l) \right] = \frac{\omega(1 - \beta^2)}{(1 - \phi)[1 - (\beta^2 + \sigma_b^2)]} \sum_{i=1}^{l-1} B^{2j} \]  
(20)
6. Estimating RCA (1)

Let \( y_n, y_{n-1}, \ldots, y_1 \) be a sample size \( n \) observations, and the RCA(1) process in (1) may be written as follows [1]:

\[
y_t = \beta y_{t-1} + v_t
\]  

(21)

Where \( v_t = b_t y_{t-1} + e_t \),

then the least squares estimate \( \hat{\beta}_{ls} \) of \( \beta \) is given by:

\[
\hat{\beta}_{ls} = \frac{\sum_{t=2}^{n}y_t y_{t-1}}{\sum_{t=2}^{n} y_{t-1}^2}
\]

(22)

After estimating \( \beta \), we may then estimate \( \sigma^2_b \) for all \( i \) and \( \sigma^2_e \) by considering equation (21) such that

\[
\hat{\sigma}_{ts}^2 = \frac{\sum_{t=2}^{n} \hat{v}_{t,ls}^2 (y_{t-1}^2 - \bar{z}^2)}{\sum_{t=2}^{n} y_{t-1}^2 - \bar{z}^2}
\]

(23)

\[
\hat{\sigma}_{b,ls}^2 = \frac{\sum_{t=2}^{n} \hat{v}_{t,ls}^2 - \hat{\sigma}_{b,ls}^2 \bar{z}}{n-1}
\]

(24)

\[
\hat{\sigma}_{e,ls}^2 = \frac{\sum_{t=2}^{n} \hat{v}_{t,ls}^2 - \hat{\sigma}_{b,ls}^2 \bar{z}}{n-1}
\]

(25)

Where \( \bar{z} = \frac{\sum_{t=2}^{n} y_{t-1}^2}{n-1} \)

7. Application

The sample representing the time series of copper’s daily financial close prices in US dollars per pound from 10/1/2010 to 29/5/2015 was approved. Figure 1 depicts the graph of the studied time series observations. It is illustrated by a graph that there is a trend with various volatilities, and then the series is nonstationary in the mean. An Augmented Dicky – Fuller test was performed, with a value (DF=2.4579) and probability of (P-value=0.1262), indicating that the series was nonstationary. Accordingly, the first difference was taken as in Figure 2, which confirms that the series becomes stationary.
After preparing the series, the random coefficient autoregressive process RCA by using the traditional least squares (LS) was estimated as follows:

<table>
<thead>
<tr>
<th>statistics</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\beta}_{ls}$</td>
<td>0.99321</td>
<td>0.10874</td>
<td>0.0001</td>
</tr>
<tr>
<td>$\hat{\sigma}_h^2$</td>
<td>0.00152</td>
<td>0.00035</td>
<td>0.0003</td>
</tr>
<tr>
<td>$\hat{\sigma}_e^2$</td>
<td>0.00231</td>
<td>0.00051</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Hence, employing a series of residuals resulting from the estimation of the RCA process to build the GARCH process. It shows that the $e_t$ series is leptokurtic ($kurt = 5.664$) and that the volatilities in the series are apparent. Therefore, the GARCH model should be applied to the residuals series of 1682 observations plotted in Figure 3.

Figures 4 and 5 show graphs of the coefficients of the (ACF) and (PACF) of the squared residuals. The ACF and PACF feature exponential decay with significant points at seasonal lags with seasonality index $s = 5$. To investigate the existence of the ARCH effect for the residuals of the RCA model, the Lagrange multiplier test was used, where the value of (LM-test=891.26) with probability (P-value = 0.000), so the null hypothesis stating that there is no effect of ARCH was rejected. Thus, the residuals series has an ARCH effect.
The residual series was also stationary since \((DF= -43.47482)\) and the probability of \((P\text{-value}=0.0001)\). A seasonal GARCH\((1,0)\times(0,1)\) model given by \((15)-(16)\) with standard normal \(Z_t\) was estimated by (MLE) for the copper data, and the results of the estimation are shown in Table 2. After evaluating a set of a running model by the maximum likelihood method when \((s=5)\), it turns out that GARCH\((1,0)\times(0,1)\) is the appropriate model.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Parameters Est.</th>
<th>Stand. Error of Estimator</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)</td>
<td>0.000277</td>
<td>0.000072</td>
<td>0.0002</td>
</tr>
<tr>
<td>(\omega)</td>
<td>5.74E-07</td>
<td>2.97E-08</td>
<td>0.0000</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.1283</td>
<td>0.00609</td>
<td>0.0000</td>
</tr>
<tr>
<td>(\theta)</td>
<td>0.8807</td>
<td>0.00487</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Finally, the Lagrange multiplier test is applied to verify the estimated model's suitability. Table 3 found that the probability of the test (P-value) was the largest (0.05), which refers to the lack of effect ARCH.

<table>
<thead>
<tr>
<th>Test</th>
<th>Value of test</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>F-statistic</td>
<td>0.053931</td>
<td>0.8164</td>
</tr>
<tr>
<td>LM-test</td>
<td>0.053994</td>
<td>0.8163</td>
</tr>
</tbody>
</table>

Then, the seasonal GARCH \((1,0)\times(0,1)\) model for the residuals is given by:

\[
h_t = \omega + \phi e_{t-1}^2 - \theta e_{t-5}^2 + \theta h_{t-5}
\]

\[
h_t = 5.74E - 07 + 0.1283 e_{t-1}^2 - 0.8807 e_{t-5}^2 + 0.8807 h_{t-5}
\]

And for comparison, A model GARCH\((1,0)\times(0,1)\) model was fitted individually for the origin data by (MLE) using two distributions of residuals, \(t_3\) and \(t_7\). And the results of the estimation are shown in Table (2) as follows

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Normal(0.1) errors</th>
<th>(t_3) errors</th>
<th>(t_7) errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters Est.</td>
<td>Stand. E. of estimators</td>
<td>Parameters Est.</td>
<td>Stand. E. of estimators</td>
</tr>
<tr>
<td>(\mu)</td>
<td>0.000277</td>
<td>0.000072</td>
<td>0.00043</td>
</tr>
<tr>
<td>(\omega)</td>
<td>5.74E-7</td>
<td>2.97E-8</td>
<td>2.77E-5</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.1283</td>
<td>0.00609</td>
<td>0.06100</td>
</tr>
<tr>
<td>(\theta)</td>
<td>0.8807</td>
<td>0.00487</td>
<td>0.95383</td>
</tr>
<tr>
<td>AIC</td>
<td>-8.0483</td>
<td>-3.4079</td>
<td>-3.4216</td>
</tr>
</tbody>
</table>

8. Conclusion

In this work, the RCA-SGARCH process is considered. The properties of moments and kurtosis of the volatility model are presented. The model studied in the paper is an example of the seasonal volatility model. High moments and kurtosis properties are also shown under the assumption of the normal distribution when \((e_t)\) and \((b_t)\) are correlated and not. An application of the daily financial close prices of copper is used. Then we made a comparison among the studied model with errors distributed as \(N(0,1)\) and the same process when the errors series were distributed as \(T\) distribution with degrees of freedom \((3)\) and \((5)\). It shows that the studied model is the best according to its parameters' significance and the Akaike information criterion (AIC) value. Also, to compare the process when the errors series are distributed as \(t_7\) and \(t_3\), the model with \(t_7\) is the better since the standard error of the parameters and AIC of it is the least.

Declaration of competing interest

There are no financial or non-financial competing interests in this paper's content, according to its authors.
Funding information
This study received no financial support from any financial institution.

References
APPENDIX

Proof for Theorems 1

\[ y_t = (\beta + b_t) y_{t-1} + e_t \]

\[ E(y_t^2) = \beta^2 E(y_{t-1}^2) + E(b_t^2 y_{t-1}^2) + 2\beta b_t E(y_{t-1}^2) + 2b_t e_t y_{t-1} + 2b_t e_t y_{t-1} + e_t^2 \]

\[ E(y_t^2) = \beta^2 E(y_{t-1}^2) + \sigma_b^2 E(y_{t-1}^2) + E(e_t^2) \]

\[ E(y_t^2) = \frac{\sigma_b^2}{[1 - \beta^2 - \sigma_b^2]} \]

\[ y_t^3 = \beta^3 y_{t-1}^3 + 3\beta^2 y_{t-1}^2 b_t + 3\beta^2 y_{t-1}^2 e_t + 3\beta y_{t-1}^3 b_t + 6\beta y_{t-1}^2 b_t e_t + 3\beta y_{t-1} e_t^2 + b_t^2 y_{t-1} + 3b_t^2 y_{t-1} e_t + 3b_t y_{t-1} e_t^2 + e_t^3 \]

\[ E(y_t^3) = \beta^2 E(y_{t-1}^3) + 3\beta \sigma_b^2 E(y_{t-1}^2) + 6\beta \rho \sigma_b \sigma_e E(y_{t-1}^2) \]

\[ E(y_t^3) = \beta^2 E(y_{t-1}^3) + 3\beta \sigma_b^2 E(y_{t-1}^2) + 6\beta \rho \sigma_b \sigma_e \frac{[1 - \beta^2 - \sigma_b^2]}{[1 - \beta^2 - \sigma_b^2]} \]

\[ E(y_t^3) = \frac{6\beta \rho \sigma_b \sigma_e^3}{[1 - \beta^2 - 3\beta \sigma_b^2] [1 - (\beta^2 + \sigma_b^2)]} \]

Then, the skewness is given by:

\[ S(y) = \frac{E(y_t^3)}{[E(y_t^2)]^{3/2}} \]

\[ = \frac{6\beta \rho \sigma_b \sigma_e^3}{[1 - \beta^2 - 3\beta \sigma_b^2] [1 - (\beta^2 + \sigma_b^2)]} \]

\[ = \frac{\frac{\sigma_e^2}{[1 - \beta^2 - \sigma_b^2]}^{3/2}}{[1 - \beta^2 - 3\beta \sigma_b^2] [1 - (\beta^2 + \sigma_b^2)]} \]

\[ = \frac{[1 - \beta^2 - 3\beta \sigma_b^2] [1 - (\beta^2 + \sigma_b^2)]}{\sigma_e^3} \]

\[ S(y) = \frac{6\beta \rho \sigma_b [1 - (\beta^2 + \sigma_b^2)]^{3/2}}{[1 - \beta^2 - 3\beta \sigma_b^2]} \]

If \( e_t \) and \( b_t \) are uncorrelated then \( \rho = 0 \), then \( S(y) = 0 \)

\[ E(y_t^4) = 6\beta^2 E(y_{t-1}^4) + 6E(b_t^2 y_{t-1}^4) + \beta^4 E(y_{t-1}^4) + E(e_t^4) + 12\beta^2 E(y_{t-1}^2) b_te_t + 6\beta^2 E(y_{t-1}^2) e_t^2 \]

\[ = 6\beta^2 \sigma_b^2 E(y_{t-1}^2) + 6\sigma_b^2 \sigma_e^2 E(y_{t-1}^2) + \beta^4 E(y_{t-1}^4) + 3\sigma_b^4 E(y_{t-1}^4) + 3\sigma_b^4 + 12\beta^2 \rho \sigma_b \sigma_e E(y_{t-1}^2) + 6\beta^2 \sigma_e^2 E(y_{t-1}^2) \]

\[ [1 - 6\beta^2 \sigma_b^2 - \beta^4 - 3\sigma_b^4] E(y_t^4) = 6\sigma_b^2 E(y_{t-1}^2) [\sigma_b^2 + \beta^2] + 12\beta^2 \rho \sigma_b \sigma_e E(y_{t-1}^2) + 3\sigma_b^4 \]

\[ [1 - 6\beta^2 \sigma_b^2 - \beta^4 - 3\sigma_b^4] E(y_t^4) = \frac{6\sigma_b^2 [\sigma_b^2 + \beta^2]}{[1 - (\beta^2 + \sigma_b^2)] [1 - (\beta^2 + \sigma_b^2)] [1 - \beta^2 - 3\beta \sigma_b^2]} + 3\sigma_b^4 \]

\[ E(y_t^4) = \left\{ \frac{6\sigma_b^2 [\sigma_b^2 + \beta^2]}{[1 - (\beta^2 + \sigma_b^2)] [1 - \beta^2 - 3\beta \sigma_b^2]} + 3\sigma_b^4 \right\} \frac{72\beta^3 \rho^2 \sigma_b \sigma_e^4}{[1 - \beta^2 - 3\beta \sigma_b^2] [1 - \beta^2 - 3\beta \sigma_b^2] + 3\sigma_b^4} \]

\[ k(y) = \left\{ \frac{6(\sigma_b^2 + \beta^2)}{[1 - \beta^3 - 3\beta \sigma_b^2]} + 72\beta^3 \rho^2 \sigma_b^2 + 3[1 - (\beta^2 + \sigma_b^2)] [1 - \beta^3 - 3\beta \sigma_b^2] \right\} \frac{[1 - \beta^3 - 3\beta \sigma_b^2]}{[1 - 6\beta^2 \sigma_b^2 - \beta^4 - 3\sigma_b^4]} \]

\[ *\left[ 1 - (\beta^2 + \sigma_b^2) \right] \]

If \( e_t \) and \( b_t \) are uncorrelated then \( \rho = 0 \), then:
\[ E(y_t^2) = \frac{3\sigma_y^4[1 + (\beta^2 + \sigma_b^2)]}{[1 - (\beta^2 + \sigma_b^2)][1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} \]

\[ k(y) = \frac{3[1 - (\beta^2 + \sigma_b^2)^2]}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} \]

Proof for equation (17)

\[
y_t^2 = \beta^2 y_{t-1}^2 + b_t^2 y_{t-1}^2 + 2\beta b_t y_{t-1}^2 + 2\beta e_{t-1} + 2b_t e_{t-1} + e_t^2
\]

\[
E(y_t^2) = \beta^2 E(y_{t-1}^2) + E(b_t^2 y_{t-1}^2) + 2\beta E(b_t y_{t-1}^2) + 2\beta E(e_{t-1}) + 2E(b_t e_{t-1}) + E(e_t^2)
\]

\[
= \beta^2 E(y_{t-1}^2) + \sigma_b^2 E(y_{t-1}^2) + E(e_t^2)
\]

\[
E(y_t^2) = \frac{6\beta^2 E(y_{t-1}^2) + 6E(b_t^2 y_{t-1}^2) + \beta^4 E(y_{t-1}^2) + 6E(b_t y_{t-1}^2) + E(e_t^2) + 6\beta E(y_{t-1}^2) e_t^2}{[1 - \beta^2 - \sigma_b^2]}
\]

\[
[1 - \beta^2 - \beta^4 - 3\sigma_b^4] E(y_t^2) = 6E(e_t^2)[\sigma_b^2 + \beta^2] E(y_{t-1}^2) + E(e_t^2)
\]

\[
[1 - \beta^2 - \beta^4 - 3\sigma_b^4] E(y_t^2) = \frac{6[\sigma_b^2 + \beta^2][E(e_t^2)]^2 + E(e_t^2)}{[1 - (\beta^2 + \sigma_b^2)]} + \frac{E(e_t^2)}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]}
\]

\[ k(y) = \frac{6[\sigma_b^2 + \beta^2][1 - (\beta^2 + \sigma_b^2)]}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} + \frac{[1 - (\beta^2 + \sigma_b^2)]^2}{[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} k(e) \]

Proof for equation (18)

From equation (7), \( \phi(B)e_t^2 = \omega + \beta(B)u_t \) where \( u_t = e_t^2 - h_t \). For any stationary process,

\[
\text{var}(e_t^2) = \sigma_u[\psi_1^2 + \psi_2^2 + \ldots]
\]

\[
\sigma_u^2 = E(e_t^2) - E(h_t^2)
\]

\[
= E(h_t^2 Z_t^2) - E(h_t^2) = E(h_t^2)E(Z_t^2) - E(h_t^2)
\]

\[
= E(h_t^2)[E(Z_t^2) - 1]
\]

\[ \therefore \text{var}(e_t^2) = E(h_t^2)[E(Z_t^2) - 1][\psi_1^2 + \psi_2^2 + \ldots] \quad (A.1) \]

But from equation (7), it follows that,

\[
\text{var}(e_t^2) = E(e_t^2) - [E(e_t^2)]^2
\]

\[
= E(h_t^2 Z_t^2) - [E(h_t)]^2
\]

\[
= E(h_t^2)E(Z_t^2) - [E(h_t)]^2
\]

Equating (A.1) and (A.2),

\[
E(h_t^2)E(Z_t^2) - [E(h_t)]^2 = E(h_t^2)[E(Z_t^2) - 1][\psi_1^2 + \psi_2^2 + \ldots]
\]

\[
E(Z_t^2) - \frac{[E(h_t)]^2}{E(h_t^2)} = E(Z_t^2) - 1[\psi_1^2 + \psi_2^2 + \ldots]
\]

\[
\frac{[E(h_t)]^2}{E(h_t^2)} = E(Z_t^2) - [E(Z_t^2) - 1][\psi_1^2 + \psi_2^2 + \ldots]
\]