Algebraic topology: On some results of topological group space

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ABSTRACT
Our main interest in this work is to study on topological group and topological groupoid spaces. We give new some results of certain types for topological group which are source proper group space denoted by (SPHG- Space), for topological groupoid are source proper topological groupoid denoted by (SPHH- Space) and for H-Space (H,π, D) are source proper groupoid space, denoted by (SPHH- Space). The important point is to get relationship of SPHГ, SPHG-Space and SPHH-Space.

Keywords: Group space, proper group space, topological group, topological groupoid

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1. Introduction
In this paper, we study SPHГ-Space, SPHG-Space and SPHH-Space and they properties for this purpose, we divide this work into section: In section one, we give the definition topological group, topological groupoid and study the properties of these spaces. In section two, we provided with several proposition about the relationship of SPHГ-Space, SPHG-Space and SPHH-Space respectively. Finally, propositions in this project are presented and debated.

2. Topological group space
In this section, we give primary concepts of this research, Let (H,D) be a topological groupoid, M be a topological Space, π: M → D be a continuous map and Let H × M denote the fiber product of α and π over D. A left action of H on (M, π, D) is a continuous map θ*: H × M → M such that (i) π(θ*(h,z)) = β(h) for each (h,z) ∈ H × M. (ii) θ*(w(π(z)), z) = z for each z ∈ M.(iii) θ*(h(θ*(h,z))) = θ*(y(h,̃h), z) for each (h,̃h) ∈ H × H and (h, z) ∈ H × M . the bundle (M,π,D) together with action θ* is called groupoid space and is denoted by H-space. [5], [6]. Let (M,π,D) be a H-space then: The subset H(Z) = {θ*(h,z)| h ∈ Hπ(z)} is the orbit of z under H. The action of H on (M,π,D) is free if for each(h,z) ∈ H × M the relation θ*(h,z) = z implies his unity. The action of H on (M,π,D) is a transitive if for each ,z) ∈ M × M , there is h ∈ H such that z = θ*(h,z) [3]. Let (M,π,D) be a H-space then: We say that (M,π,D) is free H-space if the action of H on (M,π,D) is free. We say that (M,π,D) is transitive H-space if the action of H on (M,π,D) is transitive. [2]. A topological groupoid (H,D) is called source proper groupoid (SPHG-Space) if: The source map α: H → D is a proper. The base space D is a Hansdorff. [4], [3]. A Γ – Space M is called source proper group space (SPHГ-Space) if: The action groupoid (M × Γ, M) is SPH-Space. M is free Γ – Space. [7], [1]. Let (H,D) be an SPH-Space then the α – fiber space Hx is SPHх – space for every x ∈ D. A H-space (M,π,D) is called source proper topological groupoid space(SPHH-Space) if: The action groupoid (H × M,M) is SPH-Space. (M,π,D) is free and transitive H-Space.
3. The main results of SPHG-Space, SPHG-Space and SPHH-Space

In this part, we give several properties about the relationship of SPHG-Space, SPHG-Space and SPHH-Space.

Proposition (1):
Let \((M,\pi,D)\) be H-Space and \((g \circ f, g \circ f_*)\): \((\hat{H}, \hat{D}) \to (H, D)\) be a morphism of groupoid then \((M \times \hat{D}, \hat{\pi} = P_2, \hat{D})\) is \(\hat{H}\) -Space where \(M \times \hat{D}\) is the fiber product of \(\pi\) and \(g \circ f_*\) over \(D\), \(P_2: M \times D \hat{\to} D\).

Proof:
Let \(\hat{H} \times (M \times \hat{D})\) denoted the fiber product of \(\hat{\alpha}\) and \(\hat{\pi} = P_2\) and define: \(\psi: \hat{H} \times M \times \hat{D} \to M \times \hat{D}\) by \(\psi(\hat{h}, (z, \hat{\alpha}(\hat{h}))) = (\theta^* (g \circ f(\hat{h}), z), \hat{\beta}(\hat{h})) \in M \times \hat{D}\) since \(\pi(\theta^*(g \circ f(\hat{h}), z)) = (g \circ f(\hat{h}))\) and \(\theta^*\) is a low of action of \(H\) on \((M,\pi,D)\), \(\psi\) is continuous action of \(\hat{H}\) on \((M \times \hat{D}, \hat{\pi}, \hat{\beta})\) and \(\psi\) is continuous action of \(\hat{H}\) on \((\pi, \hat{\pi}, \hat{\beta})\).

Consider the following diagram in Figure 1:

\[
\begin{array}{ccc}
\hat{H} \times (M \times \hat{D}) & \xrightarrow{\pi \circ (g \circ f)} & M \\
\downarrow \hat{D} \circ P_1 & & \downarrow \pi \\
M \times \hat{D} & \xrightarrow{\theta^* (g \circ f_*)} & M \\
\downarrow P_2 & & \downarrow \pi \\
\hat{D} & \xrightarrow{\delta^* (M \times D \hat{\to} D)} & D
\end{array}
\]

Figure 1. Diagram of Proposition (1)

In which \(\pi \circ \theta^* \circ (g \circ f) \circ P_2\) \((\hat{h}, (z, \hat{\alpha}(\hat{h}))) \to \theta^*((g \circ f)(\hat{h})(z))\) since \(\pi(\theta^*(g \circ f)(\hat{h}), z) = \hat{\beta}(g \circ f(\hat{h}))\).

Proposition (2) Let \(M\) be an SPHG-\(\Gamma\)-Space then \((M,\pi,M/\Gamma)\) is SPHG \(((M \times \Gamma) \times \hat{M}, M)\) [where\((M \times \Gamma) \times \hat{M}, M)\)] is action groupoid.

Proof: Let \(H = M \times \Gamma\) and \(H \times M\) denote the fiber product of \(M\) and \(\pi = M \to \Gamma\) over \(\Gamma\) which is the subset of \(H \times M\) of element \(\{(z, \hat{z})\}\), with \(\hat{z} = \theta(z, r)\), where \(\theta\) is a law of action of \(\Gamma\) on \(M\). Define \(\psi: H \times M \to M\) by: \(\psi(\{(z, \hat{z})\}) = \theta(z, \delta^*(\hat{z}, \hat{\Delta}))\), where \(\delta^*\) is the map \(\delta^*: (M \times D \hat{\to} D) \to \Gamma\) is continuous where \(M\).
be SPH $\Gamma$ – Space. $\psi$ is free and transitive continuous action of $H$ on $(\pi, \rho, M/\Gamma)$ since: $\pi(\psi([\hat{z}, z]), \hat{z}) = \pi(\hat{z}, \delta^*(\hat{z}, z)) = \pi(\hat{z}, \hat{z}) = \beta(\hat{z}, \hat{z}) \cdot \psi(\pi([\hat{z}, z]), \hat{z}) \cdot \psi([\hat{z}, z], \hat{z}) = \theta(z, \delta^*(z, z)) = \psi(z, z) = \pi(\hat{z}, \delta^*(\hat{z}, z_1))$$

\[
\theta(z, \delta^*(\hat{z}, z_1)) = \psi([\hat{z}, z], \theta(\hat{z}, \delta^*(\hat{z}, z_1))), \theta(\hat{z}, \delta^*(z, \theta(\hat{z}, \delta^*(\hat{z}, z_1)))
\]

Since $\pi(z) = \pi(\hat{z})$. $\Psi$ is a free action since $\psi([\hat{z}, z], z) = \pi(\hat{z}, \delta^*(\hat{z}, z)) = \pi(\hat{z}) = \pi(z)$ and then $\theta(\hat{z}, \delta^*(\hat{z}, z)) = \psi(z, z)$ is a free action since if $\psi([\hat{z}, z], z) = \theta(z, \delta^*(z, z)) = \theta(z, e) = \psi(\hat{z}, \delta^*(\hat{z}, z_1))$.

The map $\Gamma$ is free and since $\Gamma$ is compact, it is a free action.

Proposition (3): Let $M \times M / \Gamma, M / \Gamma$ be an SPH $\Gamma$ – space then $\pi$ is SPH $\pi(z) H / \pi(z) –$ Space, for all $z \in M$. Where $\pi: M \times M / \Gamma \to M / \Gamma$.

Proof: The map $\eta_z: H / \pi(z) \to M, \eta_z(h) = \theta^*(h, Z)$ where $\theta^*$ is the law of action of $H$-space and $\eta_z$ is a closed map since for every closed subset $A$ of $H / \pi(z)$ then $A$ is compact ($G / \pi(z)$ is compact) and then $\eta_z(A)$ is a homeomorphism, now define such that $\eta_z(\eta_z(h), (z)) = \eta_z(h) = \eta_z(h, (z))$ where $h$ is a unique element such that $h = \eta_z^{-1}(z)$ and $\gamma = \gamma_{\pi(z) \times \pi(z)} \pi(z)$ $\gamma_{\pi(z) \times \pi(z)}(\pi(z), \pi(z)) = \eta_z\gamma(h, w(\pi(z))) = \eta_z(\gamma(h, w(\pi(z)))) = \eta_z(h) = \hat{z}$ then $\gamma(\eta_z(h), (z)) = \eta_z(h)$ ($\eta_z$ injective) and then $\gamma$ is unity $\phi_z$.

Now to show the action groupoid $(M \times \pi(z)) H / \pi(z)$ is SPHG-Space: $M$ is Housdorff space since $H / \pi(z)$ is the composition of continuous map. Now to show the action groupoid $(M \times \pi(z)) H / \pi(z)$ is SPHG-Space: $M$ is Housdorff space since $H / \pi(z)$ is SPHG-Space. (a)The Source map $\alpha_z: M \times \pi(z) H / \pi(z) \to M, \alpha_z(\hat{z}, r) = \hat{z}$ is a proper map by using the following commutative diagram:

\[
\begin{array}{ccc}
\hat{z} & \xrightarrow{\alpha_z} & \hat{z} \\
\eta_z \times I & \downarrow & \eta_z \\
M \times \pi(z) H / \pi(z) & \xrightarrow{\alpha_z} & M
\end{array}
\]

Figure 2. Diagram of Proposition (3)

Where $\alpha_z$ is a Source map of the action groupoid $(H / \pi(z) \times \pi(z)) H / \pi(z)$ which is proper map and then we have $\alpha_z$ is proper map.

Proposition (4): Let $M$ be an SPHG-Space then
\((M \times M / \pi(Z) H_{\pi(Z)}, M / \pi(Z) H_{\pi(Z)})\) is SPH-Space for every \(z \in M\).

**Proof:**

\(M\) is \(SPH_{\pi(Z)} H_{\pi(Z)}\) - Space for every \(z \in M\) we have Ehressman groupoid \((M \times M / \pi(Z) H_{\pi(Z)}, M / \pi(Z) H_{\pi(Z)})\) is SPH-Space for every \(\in M\).

**Proposition (5):**

Let \((H, D)\) be an SPH- groupoid and \((M \times M / \pi(Z) H_{\pi(Z)})\) be SPHG-Space then the map \(\pi: M \rightarrow D\) is proper map.

**Proof:**

Consider the following commutative diagram in \(M\) as in Figure 3.

\[
\begin{array}{ccc}
\pi(Z) H_{\pi(Z)} & \xrightarrow{\eta_z} & M \\
\downarrow{\beta_{\pi(Z)}} & & \downarrow{\pi} \\
D & \xleftarrow{Id_D} & D
\end{array}
\]

Figure 3. Diagram of Proposition (4)

In which \(\eta_z\) is an homeorphism given proposition (4.1.10) and \(\beta_{\pi(Z)} = \psi|_{\pi(Z) H_{\pi(Z)}}\) is proper map since \((H, D)\) is SPHG-Space then \(\pi \circ \eta_z\) is proper map and then \(\pi\) is proper map.

4. Conclusion

New findings from certain topology groups in this paper were given, the proper grouping space denoted by \((SPH^\Gamma\text{-Space})\), the proper topological groupoid source is denoted by \((SPHG\text{-Space})\), and \(H\)-space \((H, \pi, D)\), the correct groupoid source denoted by groupoid source, indicates the correct topological groupoids \((SPHH\text{-Space})\). The point is to have \(SPH^\Gamma\), SPHG-Space and SPHH-Space associations that are mathematically important.

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**References**

