Periodicals of Engineering and Natural Sciences ISSN 2303-4521
Vol. 7, No. 3, September 2019, pp.1114-1125
Available online at: http://pen.ius.edu.ba

Bounds of the initial coefficient for sakaguchi function in the conical domain

S. Balaji[1], B. Srutha Keerthi[2]
[1,2]Department of Mathematics, School of Advanced Sciences, VIT Chennai Campus

Article Info

Received May 27, 2019

ABSTRACT

In this paper, we consider a new class of sakaguchi type functions which is defined by Ruscheweyh q-differential operator. We investigated of co-efficient inequalities and other interesting properties of this class.

Keyword:

Conical domain
Analytic functions
Sakaguchi type functions
k-uniformly starlike functions
q-differential Operator.

Corresponding Author:

B. SruthaKeerthi
Department of Mathematics, School of Advanced Sciences
VIT Chennai Campus, Chennai - 600 048, India
E-mail: sruthilaya06@yahoo.co.in

1. Introduction

Let $\mathcal{A}$ denotes the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (1)$$

which are analytic in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. If f and g are analytic function in $\mathbb{D}$, then thefunction f is said to be subordinate to g, and write $f(z)\prec g(z)$, if there exists a function ω analytic in $\mathbb{D}$ with$\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\mathbb{D}$, such that$f(z)=g(\omega(z))$, $z\in\mathbb{D}$. Moreover, if the function g isunivalent in $\mathbb{D}$, then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(\mathbb{D})\subset g(\mathbb{D})$.
Kanas and Wisniowka [5, 6], established the conic kind of domain Φ_k, $k\geq 0$ as

$$\Phi_k=\{u+iv:u>k\sqrt{(u-1)^2+v^2}\}$$

We note that Φ_k is a region in the right half-plane, symmetric with respect to real axis, and contains the point (1,0). More precisely for k=0, Φ_0 is the right half-plane, for 0<k<1, $\Phi_k$is an unbounded region having boundary $\partial\Phi_k$, a rectangular hyperbola for k=1, Φ_1 is still an unbounded region where $\partial\Phi_1$ is a parabola, and for k>1, Φ_k is a bounded region enclosed by an ellipse. The extremalfunction for these conic regions are

$$p_k(z)=\begin{cases}\dfrac{1+z}{1-z}, & k=0\\ 1+\dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1\\ \dfrac{1}{1-k^2}\cosh\left\{\left(\dfrac{2}{\pi}\operatorname{arccos}k\right)\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\dfrac{k^2}{1-k^2}, & 0<k<1\\ \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2k(\kappa k}\int_0^{\frac{u(z)}{\sqrt{\kappa}}}\dfrac{dk}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right)+\dfrac{k^2}{k^2-1}, & k>1\end{cases} \tag{2}$$

where $u(z)=\dfrac{2-\sqrt{k}}{1-\sqrt{kz}}$, $z\in\mathbb{D}$ and $k\in(0,1)$ is chosen such that $k=\cosh\left(\dfrac{\pi K'(\kappa)}{4K(\kappa)}\right)$. Here $K(\kappa)$ is Legendres's complete elliptic integral of first kind and $K'(\kappa)=K(\sqrt{1-k^2})$ and $K'(t)$ is the complementary integral of $K(t)$ for details see [1, 5, 6] and more recently [9,12,14]. If $P_k(z)=1+M_1(k)z+M_2(k)z^2+\cdots$, $z\in\mathbb{D}$, then it was shown in [6] that for (2) one can have,

$$M_1(k)=\begin{cases}\dfrac{2A^2}{1-k^2} & 0\le k<1\\ \dfrac{8}{\pi^2} & k=1\\ \dfrac{\pi^2}{4K^2(t)^2(1+t)\sqrt{t}} & k>1\end{cases} \tag{3}$$

$M_2(k)= E(k)\, M_1(k)$

where

$$E(k)=\begin{cases}\dfrac{A^2+2}{3} & 0\le k<1\\ \dfrac{8}{\pi^2} & k=1\\ \dfrac{(4K(t))^2(t^2+6t+1)-\pi^2}{24K(t)^2(1+t)\sqrt{t}} & k>1\end{cases} \tag{4}$$

with $A=\dfrac{2}{\pi}\operatorname{arccos}k$

Further more a function p is said to be in the class $k\text{–}P[A,B]$ if and only if

$$p(z)\prec q_k(z),\quad k\ge 0$$

$$\text{where}\quad q_k(z)=\frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)} \tag{5}$$

where p_k is defined in (2) and $-1\le B<A\le 1$. Geometrically the function $p\in k\text{–}P[A,B]$ takes all the values from the domain $\Phi_k[A,B]$, $-1\le B<A\le 1$, $k\ge 0$, which is defined as:

$$\Phi_k[A,B]=\{\omega:\Re((c(\omega))>k|c(\omega)|\}$$

$$\text{where}\quad c(\omega(=\left(\frac{(B-1)\omega-(A-1)}{(B+1)\omega-(A+1)}\right)$$

or equivalently $\Phi_k[A,B]$ is a set of numbers $\omega= u+iv$ such that

$$[(B^2-1)(u^2+v^2)-2(AB-1)u+(A^2-1)]^2$$
$$>k[-2(B+1)(u^2+v^2)+2(A+B+2)u-2(A+1)^2+4(A-B)^2v^2]$$

This domain represents the conic type of regions for detail see [11]. For any n positive integer n, the q-integer number n, [n,q] is defined by

$$[n,q]=\frac{1-q^{n}}{1-q}=1+q+\cdots+q^{n-1}\ \ [0,q]=0,\quad q\in(0,1) \tag{6}$$

q-differential operator be defined by

$$\partial_{q}f(z)=\frac{f(qz)-f(z)}{(q-1)z},\quad (z\in D)$$

It is easy to observe that for n $\in\mathbb{N}$:= {1,2,3...} and z $\in\mathbb{D}$

$$\partial_{q}z^{n} = [n,q]z^{n-1}$$

Let the q-generated pochhammer symbol be defined as

$$[r,q]_{n} =[r,q][r+1,q][r+2,q]...[r+n-1,q]$$

and for r>0 let the q-gamma function be defined as

$$\Gamma_{q}(r+1)=[r]\ \Gamma_{q}(r) \text{ and } \Gamma_{q}(1)=1$$

These kind of operators see [2,3,13], play great in GFT. Kanas et al, defined Rucheweyh q-differential operator as follows:

Definition 1. [7]

For the function f$\in\mathcal{A}$ is in the form (1) , the Rucheweyh q-differential operator is:

$$R_{q}^{\lambda}\ f(z)=f(z)*F_{q,\lambda+1}(z),\quad (z\in\mathbb{D},\lambda>-1) \tag{7}$$

where

$$\begin{aligned} F_{q,\lambda+1}(z)&=z+\sum_{n=2}^{\infty}\frac{\Gamma_{q}(n+\lambda)}{[n-1,q]!\Gamma_{q}(1+\lambda)}z^{n}\\ &=z+\sum_{n=2}^{\infty}\frac{[\lambda+1,q]_{n-1}}{[n-1,q]!}z^{n}\\ &=z+\sum_{n=2}^{\infty}\phi_{n-1}z^{n}\end{aligned} \tag{8}$$

where

$$\phi_{n-1}=\frac{\Gamma_{q}(n+\lambda)}{[n-1,q]!\Gamma_{q}(1+\lambda)}=\frac{[\lambda+1,q]_{n-1}}{[n-1,q]!}$$

from (7) we get that

$$R_{q}^{0}f(z)=f(z),\quad R'_{q}f(z)=z\partial_{q}f(z)$$

and

$$R_{q}^{m}f(z)=\frac{z\partial_{q}^{m}(z^{m-1}f(z))}{[m,q]!}\quad (m\in\mathbb{N})$$

Using (7) and (8), the power series $R_{q}^{\lambda}f(z)$ is given by

$$\begin{aligned} R_{q}^{\lambda}f(z)&=z+\sum_{n=2}^{\infty}\frac{\Gamma_{q}(n+\lambda)}{[n-1,q]!\Gamma_{q}(1+\lambda)}a_{n}z^{n}\\ &=z+\sum_{n=2}^{\infty}\frac{[\lambda+1,q]_{n-1}}{[n-1,q]!}a_{n}z^{n}\end{aligned} \tag{9}$$

Note that

$$\lim_{q\to1}\Gamma_{q,\lambda+1}(z)=\frac{z}{(1-z)^{\lambda+1}}$$

and

$$\lim_{q\to1}R_{q}^{\lambda}f(z)=f(z)*\frac{z}{(1-z)^{\lambda+1}}$$

When q$\to$1 see[16], we observe that

$$z\partial(F_{q,\lambda+1}(z))=\left(1+\frac{[\lambda,q]}{q^{\lambda}}\right)F_{q,\lambda+2}(z)-\frac{[\lambda,q]}{q^{\lambda}}F_{q,\lambda+1}(z) \tag{10}$$

making use of (7), (10) and the properties of hadamard product we obtain the following equality

$$z\partial(R_q^{\lambda}f(z))=\left(1+\frac{[\lambda,q]}{q^{\lambda}}\right)R_q^{\lambda+1}f(z)-\frac{[\lambda,q]}{q^{\lambda}}R_q^{\lambda}f(z) \tag{11}$$

If $q\rightarrow 1$, the equality (11) implies

$$z(R^{\lambda}f(z))'=(1+\lambda)\,R^{\lambda+1}f(z)-\lambda\,R^{\lambda}f(z)$$

which is the familiar recurrent formula for the above operator. we now defined the following classes of functions.

Definition 2.
A function $f(z)\in\mathcal{A}$ is said to be in the class $k-US_q(\lambda,A,B,t)$, $k\geq 0$, $\beta\leq 0$, $t\in\mathbb{C}$ with $|t|\leq 1$ $-1\leq B<A\leq 1$, if and only if

$$\Re\,(c(X(z)))>k\,|c(X(z))|$$

$$X(z)=\frac{(1-t)(z\partial_q R_q^{\lambda}f(z))}{(R_q^{\lambda}f(z)-R_q^{\lambda}f(tz))}$$

orequvalently,

$$X(z)=\frac{(1-t)(z\partial_q R_q^{\lambda}f(z))}{(R_q^{\lambda}f(z)-R_q^{\lambda}f(tz))}\in k-P[A,B]$$

Definition 3.
A function $f(z)\in\mathcal{A}$ is said to be in the class $k-UC_q(\lambda,A,B,t)$, $k\geq 0$, $\beta\leq 0$, $t\in\mathbb{C}$ with $|t|\leq 1$,$-1\leq B<A\leq 1$, if and only if

$$\Re\,(c(Y(z)))>k\,|c(Y(z))|$$

$$Y(z)=\frac{(1-t)(z\partial_q R_q^{\lambda}f(z)+z^2\partial_q^2 R_q^{\lambda}f(z))}{z(\partial_q R_q^{\lambda}f(z)-t\,\partial_q R_q^{\lambda}f(tz))}$$

orequvalently,

$$Y(z)=\frac{(1-t)(z\partial_q R_q^{\lambda}f(z)+z^2\partial_q^2 R_q^{\lambda}f(z))}{z(\partial_q R_q^{\lambda}f(z)-t\,\partial_q R_q^{\lambda}f(tz))}\in k-UC_q$$

Definition 4.
A function $f(z)\in\mathcal{A}$ is said to be in the class $k-US_q(\lambda,A,B,\gamma,t)$, $k\geq 0$, $\beta\leq 0$, $t\in\mathbb{C}$ with $|t|\leq 1$ $-1\leq B<A\leq 1$, if and only if

$$\Re\big(c(G(z))\big)>k\big|c(G(z))\big|$$

$$G(z)=\frac{(1-t)(z\partial_q R_q^{\lambda}f(z)+\gamma z^2\partial_q^2 R_q^{\lambda}f(z))}{(1-\gamma)(R_q^{\lambda}f(z)-R_q^{\lambda}f(tz))+\gamma z(\partial_q R_q^{\lambda}f(z)-t\partial_q R_q^{\lambda}f(tz))}$$

orequvalently,

$$G(z)=\frac{(1-t)(z\partial_q R_q^{\lambda}f(z)+\gamma z^2\partial_q^2 R_q^{\lambda}f(z))}{(1-\gamma)(R_q^{\lambda}f(z)-R_q^{\lambda}f(tz))+\gamma z(\partial_q R_q^{\lambda}f(z)-t\partial_q R_q^{\lambda}f(tz))}\in k-P[A,B] \tag{12}$$

Remark 5.
It is easily see that $\lim_{q\rightarrow 1} k-US_q(0, A, B, 0, 0) = k-ST(A, B)$ where $k-ST(A, B)$ is a functions class, investigated by Noor and sarfraz [11]

Lemma 6.[15]

Let $h(z)=1+\sum_{n=1}^{\infty}c_n z^n$ be subordinate to $H(z)=1+\sum_{n=1}^{\infty}C_n z^n$. If H(z) is univalent in $\mathbb{D}$ and H(E) is convex, then

$$|c_n| \le |C_1|,\ \ n\ge 1$$

Lemma 7. [8,10]
If $q(z)=1+c_1z+c_2z^2+\cdots$ is an analytic function with positive real part in $\mathbb{D}$ then,

$$\left|c_2 - vc_1^2\right| \le 2\max\{1,|2v-1|\}$$

The result is sharp for the function

$$q(z)=\frac{1+z^2}{1-z^2} \qquad (\text{or}) \qquad q(z)=\frac{1+z}{1-z}$$

Lemma 8. [8]
If the function $\omega\in\mathbb{D}$ is in the form

$$\omega(z)=c_1z+c_2z^2+\cdots\ z\in\mathbb{D}$$

Then,

$$\left|c_2 - vc_1^2\right| \le 1+(|v|-1)|c_1|^2$$

where v is the complex number

Lemma 9. [11]
Let $k\in[0,\infty)$ be fixed and $q_k(z)$ in the form (5)then

$$q_k(z)=1+H_1(k)z+H_2(k)z^2+\cdots,\ z\in\mathbb{D}$$

and

$$H_1 : H_1(k)=\frac{A-B}{2}M_1(k)$$

$$H_2 : H_2(k)=\frac{A-B}{4}\{2E(k)-(B+1)H_1\}M_1(k)$$

where $M_1(k)$ and E(k) are defined in (3) and (4)

2. Main Results

Theorem 10:
A function $f\in\mathcal{A}$ and of the form (1) is in the class $k\text{–}US_q(\lambda,A,B,\gamma,t)$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\{2(k+1)\{(1-\gamma)u_n+(\gamma u_n-1)[n,q]-\gamma[n,q][n-1,q]\}+|(B+1)\{[n,q]+\gamma[n,q][n-1,q]\} \\ -(A+1)\{(1-\gamma)u_n+\gamma[n,q]u_n\}|\}\phi_{n-1}|a_n| \le |B-A| \tag{13}$$

roof.
Assume (13) is hold, then it suffices to show that

$$k\left|\frac{(B-1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A-1)}{(B+1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A+1)}-1\right|$$

$$-\Re\left\{\frac{(B-1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A-1)}{(B+1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A+1)}-1\right\}<1$$

we have

$$k\left|\frac{(B-1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A-1)}{(B+1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A+1)}-1\right|$$

$$-\Re\left\{\frac{(B-1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A-1)}{(B+1)\left(\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}\right)-(A+1)}-1\right\}$$

$$\leq (k+1)\left|\frac{(B-1)(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))-(A-1)[(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))]}{(B+1)(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))-(A+1)[(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))]}-1\right|$$

$$=2(k+1)\left|\frac{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))-(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(B+1)(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))-(A+1)[(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))]}\right|$$

$$=2(k+1)\left|\frac{\sum_{n=2}^{\infty}\{(1-\gamma)u_n+(\gamma u_n-1)[n,q]-\gamma[n,q][n-1,q]\}\phi_{n-1}a_n z^n}{(B-A)z+\sum_{n=2}^{\infty}\{(B+1)\{[n,q]+\gamma[n,q][n-1,q]\}-(A+1)\{(1-\gamma)u_n+\gamma[n,q]u_n\}\}\phi_{n-1}a_n z^n}\right|$$

$$=2(k+1)\frac{\sum_{n=2}^{\infty}\{(1-\gamma)u_n+(\gamma u_n-1)[n,q]-\gamma[n,q][n-1,q]\}\phi_{n-1}|a_n|}{|B-A|-\sum_{n=2}^{\infty}\{(B+1)\{[n,q]+\gamma[n,q][n-1,q]\}-(A+1)\{(1-\gamma)u_n+\gamma[n,q]u_n\}\}\phi_{n-1}|a_n|}$$

< 1 by (13) □

When $q\to 1$ and $\gamma = 0$ we have,

Corollary 11.
A function $f \in A$ and of the form (1) is in the class $k\text{–}US_q(\lambda,A,B,t)$, if it satisfies the condition,

$$\sum_{n=2}^{\infty}\{2(k+1)(u_n-n)+|n(B+1)-u_n(A+1)|\}\phi_{n-1}|a_n|\leq|B-A|$$

When $q\to 1$ and $\gamma = 1$ we have,

Corollary 12.
A function $f \in A$ and of the form (1) is in the class $k\text{–}UC_q(\lambda,A,B,t)$, if it satisfies the condition,

$$\sum_{n=2}^{\infty}\{2(k+1)\{n(u_n-1)-n(n-1)\}+\left|(B+1)n^2-nu_n(A+1)\right|\}\phi_{n-1}|a_n|\le|B-A|$$

Theorem 13.
If $f(z)\in k\text{–}US_q(\lambda,A,B,\gamma,t)$, and is of the form (1). then

$$|a_n|\le\prod_{j=0}^{n-2}\left\{\frac{\left|M_1(A-B)u_{j+1}\{(1-\gamma)+\gamma[j+1,q]\}-2[\{1+\gamma[j,q]\}[j+1,q]-\{(1-\gamma)+\gamma[j+1,q]\}u_{j+1}]B\right|}{2[\{1+\gamma[j+1,q]\}[j+2,q]-\{(1-\gamma)+\gamma[j+2,q]\}u_{j+2}]\phi_{n-1}}\right\},n\ge 2 \tag{14}$$

where $M_1(k)$ is defined by (3)

Proof.
Let

$$\frac{(1-t)(z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))}=p(z) \tag{15}$$

then

$$\begin{aligned}p(z)&\prec q_k(z)\\&=[(A+1)p_k(z)-(A-1)][(B+1)p_k(z)-(B-1)]^{-1}\\&=\frac{(A-1)}{(B-1)}+\left(\frac{(A-1)(B+1)}{(B-1)^2}-\frac{(A+1)}{(B-1)}\right)(p_k(z))+\left(\frac{(A-1)(B+1)^2}{(B-1)^3}-\frac{(A+1)(B+1)}{(B-1)^2}\right)(p_k(z))^2+\cdots\end{aligned}$$

By taking $p_k(z)=1+M_1(k)z+M_2(k)z^2+\cdots$
after some simplification, we obtain,

$$p(z)\prec\sum_{n=1}^{\infty}\frac{-2(B+1)^{n-1}}{(B-1)^n}+\left\{\sum_{n=1}^{\infty}\frac{-2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}\right\}M_1(k)+\cdots$$

Now we see that the series $\sum_{n=1}^{\infty}\frac{-2(B+1)^{n-1}}{(B-1)^n}$ and $\sum_{n=1}^{\infty}\frac{-2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}$

are convergent and converge to 1 and $\frac{A-B}{2}$ respectively. Therefore,

$$p(z)\prec 1+\frac{A-B}{2}M_1(k)z+\cdots$$

Now if $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$, then by lemma [6], we have

$$|c_n|\le\frac{A-B}{2}M_1(k),\quad n\ge 1 \tag{16}$$

Now from (15) we have

$$(1-t)[z\partial_q R_q^\lambda f(z)+\gamma z^2\partial_q^2 R_q^\lambda f(z)]=[(1-\gamma)(R_q^\lambda f(z)-R_q^\lambda f(tz))+\gamma z(\partial_q R_q^\lambda f(z)-t\partial_q R_q^\lambda f(tz))]p(z)$$

by simple calculation provides us,

$$|a_n|\le\frac{\sum_{j=1}^{n-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|\,|c_{n-j}|}{[\{1+\gamma[n-1,q]\}[n,q]-\{(1-\gamma)+\gamma[n,q]\}u_n]\phi_{n-1}},\quad a_1=1$$

$$|a_n|\le\frac{(A-B)|M_1(k)|\sum_{j=1}^{n-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|}{2[\{1+\gamma[n-1,q]\}[n,q]-\{(1-\gamma)+\gamma[n,q]\}u_n]\phi_{n-1}},\quad a_1=1 \tag{17}$$

Now we prove that

$$\frac{(A-B)|M_1(k)|\sum_{j=1}^{n-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|}{2[\{1+\gamma[n-1,q]\}[n,q]-\{(1-\gamma)+\gamma[n,q]\}u_n]\phi_{n-1}} \le \prod_{j=0}^{n-2}\left\{\frac{|M_1(k)(A-B)u_{j+1}\{(1-\gamma)+\gamma[j+1,q]\}-2[\{1+\gamma[j,q]\}[j+1,q]-\{(1-\gamma)+\gamma[j+1,q]\}u_{j+1}]B|}{2[\{1+\gamma[j+1,q]\}[j+2,q]-\{(1-\gamma)+\gamma[j+2,q]\}u_{j+2}]\phi_{n-1}}\right\} \quad (18)$$

For this we use the induction method
For n=2 From (17), we have

$$|a_2| \le \frac{(A-B)|M_1(k)|}{2[(1+\gamma)[2,q]-\{(1-\gamma)+\gamma[2,q]\}u_2]\phi_1}$$

From (14) we have

$$|a_2| \le \frac{(A-B)|M_1(k)|}{2[(1+\gamma)[2,q]-\{(1-\gamma)+\gamma[2,q]\}u_2]\phi_1}$$

For n=3 From (17) we have

$$|a_3| \le \frac{(A-B)|M_1|}{2[\{1+\gamma[2,q]\}[3,q]-\{(1-\gamma+\gamma[3,q])\}u_3]\phi_2}\left(\frac{|M_1(k)|(A-B)u_2[(1-\gamma)+\gamma[2,q]]+2[(1+\gamma)[2,q]-\{(1-\gamma)+\gamma[2,q]\}u_2]}{2[(1+\gamma)[2,q]-\{(1-\gamma)+\gamma[2,q]\}u_2]}\right)$$

from (14) we have

$$|a_3| \le \frac{(A-B)|M_1|}{2[(1+\gamma)[2,q]-\{(1-\gamma)+\gamma[2,q]\}u_2]\phi_2}\left(\frac{|M_1(k)|(A-B)u_2[(1-\gamma)+\gamma[2,q]]+2[(1+\gamma)[2,q]-\{(1-\gamma)+\gamma[2,q]\}u_2]}{2[\{1+\gamma[2,q]\}[3,q]-\{(1-\gamma)+\gamma[3,q]\}u_3]}\right)$$

Let the hypothesis be true for n=m from (16) we have,

$$|a_m| \le \frac{(A-B)|M_1(k)|\sum_{j=1}^{m-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|}{2[\{1+\gamma[m-1,q]\}[m,q]-\{(1-\gamma)+\gamma[m,q]\}u_m]\phi_{m-1}}, \quad a_1=1$$

From(14) we have

$$|a_m| \le \prod_{j=0}^{m-2}\left\{\frac{|M_1|(A-B)u_{j+1}\{(1-\gamma)+\gamma[j+1,q]\}+2[\{1+\gamma[j,q]\}[j+1,q]-\{(1-\gamma)+\gamma[j+1,q]\}u_{j+1}]}{2[\{1+\gamma[j+1,q]\}[j+2,q]-\{(1-\gamma)+\gamma[j+2,q]\}u_{j+2}]\phi_{m-1}}\right\}$$

By induction hypothesis, we have

$$\frac{(A-B)|M_1(k)|\sum_{j=1}^{m-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|}{2[\{1+\gamma[m-1,q]\}[m,q]-\{(1-\gamma)+\gamma[m,q]\}u_m]\phi_{m-1}} \le \prod_{j=0}^{m-2}\left\{\frac{|M_1|(A-B)u_{j+1}\{(1-\gamma)+\gamma[j+1,q]\}+2[\{1+\gamma[j,q]\}[j+1,q]-\{(1-\gamma)+\gamma[j+1,q]\}u_{j+1}]}{2[\{1+\gamma[j+1,q]\}[j+2,q]-\{(1-\gamma)+\gamma[j+2,q]\}u_{j+2}]\phi_{m-1}}\right\} \quad (19)$$

Multiplying both sides (19) by

$$\frac{|M_1|(A-B)u_m\{(1-\gamma)+\gamma[m,q]\}+2[\{1+\gamma[m-1,q]\}[m,q]-\{(1-\gamma)+\gamma[m,q]\}u_m]}{2[\{1+\gamma[m,q]\}[m+1,q]-\{(1-\gamma)+\gamma[m+1,q]\}u_{m+1}]}$$

we have

$$\prod_{j=0}^{m-2}\left\{\frac{|M_1|(A-B)u_{j+1}\{(1-\gamma)+\gamma[j+1,q]\}+2[\{1+\gamma[j,q]\}[j+1,q]-\{(1-\gamma)+\gamma[j+1,q]\}u_{j+1}]}{2[\{1+\gamma[j+1,q]\}[j+2,q]-\{(1-\gamma)+\gamma[j+2,q]\}u_{j+2}]\phi_{m-1}}\right\}$$

$$\geq \frac{|M_1|(A-B)u_m\{(1-\gamma)+\gamma[m,q]\}+2[\{1+\gamma[m-1,q]\}[m,q]-\{(1-\gamma)+\gamma[m,q]\}u_m]}{2[\{1+\gamma[m,q]\}[m+1,q]-\{(1-\gamma)+\gamma[m+1,q]\}u_{m+1}]}$$

$$\times\left\{|a_m| \leq \frac{(A-B)|M_1(k)|\sum_{j=1}^{m-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|}{2[\{1+\gamma[m-1,q]\}[m,q]-\{(1-\gamma)+\gamma[m,q]\}u_m]\phi_{m-1}}\right\}$$

$$=\frac{(A-B)|M_1|}{2[\{1+\gamma[m,q]\}[m+1,q]-\{(1-\gamma)+\gamma[m+1,q]\}u_{m+1}]\phi_{m-1}}$$

$$\left[\sum_{j=1}^{m-1}[(1-\gamma)+\gamma[j,q]]\phi_{j-1}|a_j|+u_m\{(1-\gamma)+\gamma[m,q]\}\phi_{m-1}|a_m|\right]$$

$$=\frac{(A-B)|M_1|}{2[\{1+\gamma[m,q]\}[m+1,q]-\{(1-\gamma)+\gamma[m+1,q]\}u_{m+1}]\phi_{m-1}}$$

$$\times\sum_{j=1}^{m}[(1-\gamma)+\gamma[j,q]]\phi_{j-1}|a_j|$$

That is

$$\frac{(A-B)|M_1(k)|\sum_{j=1}^{m-1}[(1-\gamma)+\gamma[j,q]]u_j\phi_{j-1}|a_j|}{2[\{1+\gamma[m,q]\}[m+1,q]-\{(1-\gamma)+\gamma[m+1,q]\}u_{m+1}]\phi_{m-1}}$$

$$\leq \prod_{j=0}^{m-2}\left\{\frac{|M_1|(A-B)u_{j+1}\{(1-\gamma)+\gamma[j+1,q]\}+2[\{1+\gamma[j,q]\}[j+1,q]-\{(1-\gamma)+\gamma[j+1,q]\}u_{j+1}]}{2[\{1+\gamma[j+1,q]\}[j+2,q]-\{(1-\gamma)+\gamma[j+2,q]\}u_{j+2}]\phi_{m-1}}\right\}$$

Which gives (19). □

When $q \to 1$ and $\gamma = 0$ we have,

Corollary 14.
A function $f \in A$ and of the form (1) is in $k\text{–}US_q(\lambda,A,B,t)$, if,

$$|a_n| \leq \prod_{j=0}^{n-2}\left(\frac{|M_1(k)(A-B)u_{j+1}-2[j+1-u_{j+1}]B|}{2[j+2-u_{j+2}]\phi_{n-1}}\right)$$

is satisfied.

When $q \to 1$ and $\gamma = 1$ we have,

Corollary 15.
A function $f \in A$ and of the form (1) is said to be in the class $k\text{–}UC_q(\lambda,A,B,t)$, if,

$$|a_n| \leq \prod_{j=0}^{n-2}\left(\frac{|M_1(k)(A-B)(j+1)-2(j+1)[1+j-u_{j+1}]B|}{2(j+2)[2+j-u_{j+2}]\phi_{n-1}}\right)$$

is true.

Theorem 16.
Let $-1 \leq B < A \leq 1$ and $t=0$, $0 \leq k < \infty$ be fixed and let $f(z) \in k\text{–}US_q(\lambda,A,B,\gamma,t)$ and is of the form (1) then for a complex number μ

$$|a_3-\mu a_2^2| \leq \begin{cases} \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\left|2+\frac{2E(k)-(1+B)M_1(k)}{2}+\frac{(A-B)M_1(k)}{2\{P-Ru_2\}}\left(Ru_2-\frac{\mu\{Q-Su_3\}\phi_2}{\{P-Ru_2\}(\phi_1)^2}\right)\right|, & (\mu > \delta_1) \\ \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2} & (\delta_1 \leq \mu \leq \delta_2) \\ \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\left[\frac{2E(k)-(1+B)M_1(k)}{2}+\frac{(A-B)M_1(k)}{2\{P-Ru_2\}}\left(Ru_2-\frac{\mu\{Q-Su_3\}\phi_2}{\{P-Ru_2\}(\phi_1)^2}\right)\right] & (\mu < \delta_2) \end{cases} \quad (20)$$

where

$$\delta_1 = \frac{(\phi_1)^2\{P - Ru_2\}}{\phi_2(A - B)M_1(k)\{Q - Su_3\}}\{\{2 + 2E(k) - (1 + B)M_1(k)\}\{P - Ru_2\} + (A - B)M_1(k)Ru_2\} \quad (21)$$

$$\delta_2 = \frac{(\phi_1)^2\{P - Ru_2\}}{\phi_2(A - B)M_1(k)\{Q - Su_3\}}\{\{2E(k) - (1 + B)M_1(k) - 2\}\{P - Ru_2\} + (A - B)M_1(k)Ru_2\} \quad (22)$$

and

$P=(1+\gamma)[2,q]$ $Q=(1+\gamma[2,q])[3,q]$

$R= (1-\gamma)+\gamma[2,q]$ $S=(1-\gamma)+\gamma[3,q]$

and $M_1(k)$, $E(k)$ are defined in (3) and (4)

Proof.

If $f(z) \in k-US_q(\lambda,A,B,\gamma,t)$ then it follows that

$$\frac{(1-t)(z\partial_q R_q^\lambda f(z) + \gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z) - R_q^\lambda f(tz)) + \gamma z(\partial_q R_q^\lambda f(z) - t\partial_q R_q^\lambda f(tz))} \prec q_k(z)$$
$$= 1 + \frac{(A-B)}{2}M_1(k)z + \frac{\{2E(k) - (1+B)M_1(k)\}(A-B)}{4}M_1(k)z^2 + \cdots \quad (23)$$

Now by the definition of subordination there exists a function ω analytic in $\mathbb{D}$ with $\omega(0)=0$ and $|\omega(z)|<1$ such that

$$\frac{(1-t)(z\partial_q R_q^\lambda f(z) + \gamma z^2\partial_q^2 R_q^\lambda f(z))}{(1-\gamma)(R_q^\lambda f(z) - R_q^\lambda f(tz)) + \gamma z(\partial_q R_q^\lambda f(z) - t\partial_q R_q^\lambda f(tz))}$$
$$= 1 + \frac{(A-B)}{2}M_1(k)\omega(z) + \frac{\{2E(k) - (1+B)M_1(k)\}(A-B)}{4}M_1(k)\omega^2(z) + \cdots \quad (24)$$

Now from lemma 8, equation (23) and equation (24), We have

$$a_2 = \frac{(A-B)M_1(k)c_1}{2[P - Ru_2]\phi_1}$$

and

$$a_3 = \frac{(A-B)M_1(k)}{2[Q - Su_3]\phi_2}\left\{c_2 + \left(\frac{[2E(k) - (1+B)M_1(k)]}{2} + \frac{Ru_2(A-B)M_1(k)}{2[P - Ru_2]}\right)c_1^2\right\}$$

$$\left|a_3 - \mu a_2^2\right| = \frac{(A-B)M_1(k)}{2\{Q - Su_3\}\phi_2}\left|c_2 + \left\{\frac{2E(K) - (1+B)M_1(k)}{2} + \frac{(A-B)M_1(k)}{2\{P - Ru_2\}}\left(Ru_2 - \frac{\mu\{Q - Su_3\}\phi_2}{\{P - Ru_2\}(\phi_1)^2}\right)\right\}c_1^2\right|, \quad (25)$$

which gives

$$\left|a_3 - \mu a_2^2\right| = \frac{(A-B)M_1(k)}{2\{Q - Su_3\}\phi_2}\left|c_2 - c_1^2 + \left\{1 + \frac{2E(K) - (1+B)M_1(k)}{2} + \frac{(A-B)M_1(k)}{2\{P - Ru_2\}}\left(Ru_2 - \frac{\mu\{Q - Su_3\}\phi_2}{\{P - Ru_2\}(\phi_1)^2}\right)\right\}c_1^2\right|, \quad (26)$$

Suppose that $\mu>\delta_1$ then using the estimates $\left|c_2 - c_1^2\right| \le 1$ from lemma 8 and the well known estimate $|c_1| \le 1$ of the Schewarz lemma, we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)M_1(k)}{2\{Q - Su_3\}\phi_2}\left|2 + \frac{2E(k) - (1+B)M_1(k)}{2} + \frac{(A-B)M_1(k)}{2\{P - Ru_2\}}\left(Ru_2 - \frac{\mu\{Q - Su_3\}\phi_2}{\{P - Ru_2\}(\phi_1)^2}\right)\right|, \quad (27)$$

The inequality (27) is our required assertion (20) for $\mu>\delta_1$ on other hand if $\mu<\delta_2$ then (25) gives,

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)M_1(k)}{2\{Q - Su_3\}\phi_2}\left[\left|c_2\right| + \left\{\frac{2E(k) - (1+B)M_1(k)}{2} + \frac{(A-B)M_1(k)}{2\{P - Ru_2\}}\left(Ru_2 - \frac{\mu\{Q - Su_3\}\phi_2}{\{P - Ru_2\}(\phi_1)^2}\right)\right\}|c_1^2|\right],$$

(28)

Applying the estimates $|c_2| \le 1-|c_1|^2$ of lemma 8 and $|c_1| \le 1$, We have

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\left[\frac{2E(K)-(1+B)M_1(k)}{2}+\frac{(A-B)M_1(k)}{2\{P-Ru_2\}}\left(Ru_2-\frac{\mu\{Q-Su_3\}\phi_2}{\{P-Ru_2\}(\phi_1)^2}\right)\right]$$

This is last inequality in (20). Finally if $\delta_1<\mu<\delta_2$ then

$$\left|\frac{2E(k)-(1+B)M_1(k)}{2}+\frac{(A-B)M_1(k)}{2\{P-Ru_2\}}\left(Ru_2-\frac{\mu\{Q-Su_3\}\phi_2}{\{P-Ru_2\}(\phi_1)^2}\right)\right| \le 1$$

Therefore (25) yields

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\{|c_2|+|c_1|^2\}$$

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\{1-|c_2|+|c_1|^2\}$$

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}$$

We get the middle inequality in(20). This completes the proof. □

Theorem 17.
Let $\le k < \infty$, $-1 \le B < A \le 1$ and t=0 be fixed and let $f(z) \in k-US_q(\lambda, A, B, \gamma, t)$ and is of the form (1). Then for a complex number μ

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)|M_1(k)|}{\{Q-Su_3\}\phi_2}\max\{1,|2v-1|\}$$

where v is given by (31)

Proof.
From (25) we have

$$\left|a_3-\mu a_2^2\right| = \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\left|c_2-\left\{\frac{(1+B)M_1(k)-2E(k)}{2}-\frac{(A-B)M_1(k)}{2\{P-Ru_2\}}\left(Ru_2-\frac{\mu\{Q-Su_3\}\phi_2}{\{P-Ru_2\}(\phi_1)^2}\right)\right\}c_1^2\right|, \tag{29}$$

$$= \frac{(A-B)M_1(k)}{2\{Q-Su_3\}\phi_2}\left|c_2-vc_1^2\right| \tag{30}$$

where

$$v = \frac{(1+B)M_1(k)-2E(k)}{2}-\frac{(A-B)M_1(k)}{2\{P-Ru_2\}}\left(Ru_2-\frac{\mu\{Q-Su_3\}\phi_2}{\{P-Ru_2\}(\phi_1)^2}\right) \tag{31}$$

Applying the lemma 7 on the equation (30), we obtain the required result.

□

References

[1] N.I. Ahiezer, "Elements of theory of elliptic functions", Moscow, 1970.

[2] S. Hussain, S. Khan, M.A. Zaighum and M.Darus, "Certain subclass of analytic functions related with conic domains and associated with salagean q-differential operator", *AIMS Math.*,vol.2, no. 4, pp. 622-634, 2017.

[3] S.Hussain, S. Khan, M.A. Zaighum, M.Darus and Z.Shareef, "Coefficient Bounds for Certain Subclass of Biunivalent Functions Associated with Rucheweyh q-differential Operator", *J. Complex Anal.*, 2017 (2017).

[4] W. Janowski, “Some extremal problems for certain families of analytic functions”, *Ann. Polon.Math.*, vol. 28, pp. 297-326, 1973.

[5] S. Kanas and A.Wisniowska, “Conic regions and k-uniform convexity”, *J. Comput. Appl. Math.*,vol. 105, pp. 327-336, 1999.

[6] S. Kanas and A. Wisniowska, “Conic domains and starlike functions”, *Rev. RoumaineMath.Pures Appl.*, vol. 45, pp. 647-657, 2000.

[7] S.Kanas and D. Raducanu, “Some classes of analytic functions related to conic domains”, Math.slovaca, vol. 64, no. 5, pp. 1183-1196, 2014.

[8] F.R. Keogh and E.P. Merkes, “A coefficient inequality for certain classes of analytic functions”, *Proc. Amer. Math. Soc.*, vol. 20, pp. 8-12, 1969.

[9] N. Khan, B. Khan, Q.Z. Ahmad and S.Ahmad, “Some Convolution properties of Multivalent Analytic Functions”,*AIMS Math.*, vol. 2, no. 2, pp. 260-268, 2017.

[10] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions. In: *Proc. of the Conference on Complex Analysis* (Tianjin), 1992 (Z. Li, F.Y.Ren, L.Yang, S.Y. Zhang,eds.), *Conf. Proc.Lecture Notes Anal.*, Int. Press, Massachusetts, vol 1, pp. 157-169, 1994.

[11] K.I. Noor and S.N. Malik, “On coefficient inequalities of functions associated with conic domains”, *Comput. Math. Appl.*, vol. 62, pp. 2209-2217, 2011.

[12] K.I. Noor, J. Sokol and Q.Z. Ahmad, “Applications of conic type regions to subclasses of meromorphicunivalent functions with respect to symmetric points”, *Rev. R. Acad. Cienc. Exactas Fs.Nat., Ser. A Mat.*,vol. 111, pp. 947C958, 2017.

[13] K.I. Noor, J. Sokol and Q.Z. Ahmad, “Applications of the differential operator to a class of meromorphic univalent functions”,*J. Egyptian Math. Soc.*,vol. 24, no. 2, pp. 181-186, 2016.

[14] M. Nunokawa, S. Hussain, N.Khan and Q.Z. Ahmad, “A subclass of analytic functions related with conic domain”, *J. Clas. Anal.*,vol. 9, pp. 137-149, 2016.

[15] W.Rogosinski, “On the coefficient of subordinate functions”, *Proc. Lond. Math. Soc.*, vol. 48, pp. 48-82, 1943.

[16] S.T. Rucheweyh, “New criteria for univalent functions”, *Proc. Amer. Math. Soc.*, vol. 49, pp. 109-115, 1975.

[17] H. Silverman, “Univalent functions with negative coefficient”, *Proc. Amer. Math. Soc.*, vol. 51, pp. 109-116, 1975.

[18] S. Shams, S. R. Kulkarni and J.M. Jahangiri, “Classes of uniformly starlike and convex functions”, *Int. J. Math. Sci.*, vol. 55, pp. 2959-2961, 2004.

[19] S. Khaan, S. Hussain, M.S. Zhaighum and M. Mumtaz Khan, “New subclass of analytic functions in conical domain associated with Rucheweyh q-differential operator”, *IJAA*, vol. 16, no. 2, pp. 239-253, 2018.

[20] G. Saravanan, Muthunagai. K, “Coefficient Estimates and Fekete- Szegő Inequality for a Subclass of Bi-Univalent Functions Defined by Symmetric Q-Derivative Operatorby Using Faber Polynomial Techniques” , ” Periodicals of Engineering and Natural Sciences, Vol.6, No.1, June 2018, pp. 241~250.

[21] N.P. Damodaran , SruthaKeerthi.B, “Coefficient bounds for a subclass of Sakaguchi type functions using Chebyshev Polynomial”, ” Periodicals of Engineering and Natural Sciences, Vol.6, No.1, June 2018, pp. 296-304.

[22] P. Murugabharathi, B. SruthaKeerthi , **“**Designing Filter for Certain Subclasses ofAnalytic Univalent Functions”, ” Periodicals of Engineering and Natural Sciences,Vol.6, No.1, June 2018, pp. 274~284

[23] Migdat I. Hodžić ”Uncertainty Balance Principle”PERIODICALS OF ENGINEERING AND NATURAL SCIENCES Vol. 4 No. 2 (2016).

[24] Narayanan Venkateswaran “Efficient read monotonic data aggregation across shards on the cloud” Periodicals of Engineering and Natural Sciences Vol.7, No.1, June 2019, pp.125-140.