

Estimating systems reliability functions for the generalized exponential distribution with application

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ABSTRACT

This article dealt with estimating and analyzing the reliability function for hybrid systems. The hybrid systems included series-parallel and parallel-series with symmetric components representing the machine's operation times; these times follow a generalized exponential distribution. Furthermore, we suggested employing a hybrid system parallel series in the application part of this research, which includes the operation times of Baghdad's Vegetable Oil factory machines. The results of the reliability function for the factory system indicate that the probability that the factory machines will stop during the first hour is more than 40%. Still, if the hybrid parallel-series system is applied, the probability that the factory operating system will not stop during the second hour becomes 80%.

Keywords: Reliability function, Hybrid systems, Generalized Exponential Distribution.

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1. Introduction

An ever-increasing interest in studying the causes of disruptions of all types of devices and machines has been imposed due to the growth of industry and the rise in mechanical, electrical, and electronic complexities in devices and equipment over the past century, as well as rapid technological developments and the use of complex systems in a wide range of contexts. As a result, there is a greater appetite for learning about dependability and reliability systems. The study by Al-Nidawi [2] on estimating the reliability function of hybrid parallel-series systems is one of the few that has dealt with the estimate of reliability functions for hybrid systems. While Al-Saady [3] modelled the reliability function of a hybrid series-parallel system for symmetric components, we are unaware of any other research that has attempted to estimate the reliability function of a system whose parts follow a generalized exponential distribution. Generalized Exponential Distribution (GED) is a popular method for analyzing survival and reliability data. It was first described by Gupta and Kundu (1999) [6] as a particular example of the three-parameter generalized Weibull distribution. Amongy (2020) [1], Kundu and Pradhan [8], and other recent publications do a nice job of studying inferences about (GED) parameters using various methods. The General Company for Vegetable Oil Industry has been plagued by a seemingly impossible amount of machine breakdowns. This study will illuminate this issue and provide a dependable solution well-suited to the needs of this factory. To do so, the reliability function for the following machines' failure times was estimated using the maximum likelihood approach, as was the reliability function of series-parallel and parallel-series hybrid systems.

2. Methods

2.1. Generalized exponential distribution

The p. d. f for GED random variable is defined as [3, 5, 7, 8, 9]:

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$$f(t, \mu, \beta) = \mu \beta (1 - e^{-\beta t})^{\mu-1} e^{-\beta t}, \mu, \beta > 0, t \geq 0 \tag{1}$$

Where μ and β are the shape and scale parameters, respectively.

The cumulative density function c. d. f. is defined mathematically according to the following formula:

$$F(t, \mu, \beta) = (1 - e^{-\beta t})^\mu, \mu, \beta > 0, t \geq 0 \tag{2}$$

Besides, the Reliability Function of a random time variable that follows the G.E. distribution will be according to the following formula:

$$R(t, \mu, \beta) = 1 - (1 - e^{-\beta t})^\mu, \mu, \beta > 0, t \geq 0 \tag{3}$$

For a random sample (t_1, t_2, \dots, t_n) from (GED) the Likelihood estimators for (μ) and (β) can be found as follows:

$$L(t_i, \mu, \beta) = \mu^n \beta^n e^{-\beta \sum_{i=1}^n t_i} \prod_{i=1}^n (1 - e^{-\beta t_i})^{\mu-1} \tag{4}$$

$$\ln L(t_i, \mu, \beta) = n \ln(\mu) + n \ln(\beta) + (\mu - 1) \sum_{i=1}^n \ln(1 - e^{-\beta t_i}) - \beta \sum_{i=1}^n t_i \tag{5}$$

The first derivative of both sides of equation (5) concerning (μ) and (β) , then equaling the results to zero, we obtain the MLE of (μ) as a function of (β)

$$\hat{\mu}(\beta) = -\frac{n}{\sum_{i=1}^n \ln(1 - e^{-\beta t_i})} \tag{6}$$

putting $(\mu) = \hat{\mu}(\beta)$ in (5) to get:

$$f(\beta) = A - n \ln \sum_{i=1}^n (1 - e^{-\beta t_i}) + n \ln(\beta) - \sum_{i=1}^n \ln(1 - e^{-\beta t_i}) - \beta \sum_{i=1}^n t_i \tag{7}$$

Where A is a constant, $A = [n \ln(n) - n]$.

Therefore, by maximizing (7) concerning β , we can obtain the MLE of (β) , say $(\hat{\beta}_{ML})$. Gupta and Kundu proved that the function $f(\beta) = \ln L[\hat{\mu}(\beta), \beta]$ is a unimodal function, the value of $(\hat{\beta}_{ML})$, which makes equation (7) as greatest as possible, can be obtained from the fixed points of the blow solution:

$$g(\beta) = \beta = \left[\frac{\sum_{i=1}^n \frac{t_i e^{-\beta t_i}}{1 - e^{-\beta t_i}}}{\sum_{i=1}^n \ln(1 - e^{-\beta t_i})} + \frac{1}{n} \sum_{i=1}^n \frac{t_i}{1 - e^{-\beta t_i}} \right]^{-1} \tag{8}$$

Then we can get the MLE for (μ) say $(\hat{\mu}_{ML})$ form (6) as $\hat{\mu}_{ML} = \hat{\mu}(\hat{\beta}_{ML})$. Also, the reliability function can be obtained based on the invariant property, i.e. by substituting $\hat{\mu}_{ML}$ and $\hat{\beta}_{ML}$ in (3), thus:

$$R(t) = 1 - (1 - e^{-\hat{\beta}_{ML} t})^{\hat{\mu}_{ML}} \tag{9}$$

2.2. Reliability of systems

The reliability function of systems is defined as an indicator of the probability that the System will not fail during the period [0, t]. Suppose any system contains several components, then it is necessary to determine the link type between those components; based on that, the systems can be classified into two following types [2, 3, 10]:

The series system components are connected respectively, and the failure of any component causes system failure. Let the System consists of (m) components, and then the reliability function of the series system can be expressed mathematically as follows:

$$RS(t) = R_1(t).R_2(t) \dots R_m(t) = \prod_{j=1}^m R_j(t) \quad , \quad j = 1,2, \dots, m \quad (10)$$

Parallel System is a system in which constituents are connected such that the System does not fail if any component fails. The reliability of the parallel system mathematical formula is given as follows:

$$RP(t) = 1 - \prod_{j=1}^m [1 - R_j(t)] \quad , \quad j = 1,2, \dots, m \quad (11)$$

Now if a system consists of many partial systems, such that each partial System contains several components within each partial System, and if these partial systems are linked together in parallel or series, here we get a hybrid system. Generally, there are the following two types of hybrid systems:

Hybrid Parallel - Series System is a hybrid system containing (b) partial series systems series; within each partial System, there are (m) components connected in parallel. The reliability function for a hybrid parallel-series system, say (RHPS), can be calculated as:

$$RHPS(t) = \prod_{k=1}^b \left[1 - \prod_{j=1}^m \{1 - R_{kj}(t)\} \right], k = 1,2, \dots, b, j = 1,2, \dots, m \quad (12)$$

For j=4, k=2, we have:

$$\begin{aligned} RHPS(t) &= \prod_{k=1}^2 \left[1 - \prod_{j=1}^4 \{1 - R_{kj}(t)\} \right] \\ RHPS(t) &= [1 - \{1 - R_{1A}(t)\}\{1 - R_{1B}(t)\}\{1 - R_{1C}(t)\}\{1 - R_{1D}(t)\}] * \\ & [1 - \{1 - R_{2A}(t)\}\{1 - R_{2B}(t)\}\{1 - R_{2C}(t)\}\{1 - R_{2D}(t)\}] \end{aligned} \quad (13)$$

Suppose a (HPS) of four components (A to D), each component has (GED), here j=4, k=2, then MLE for the reliability function of this System say $\hat{R}HPS(t)$ can be computed as:

$$\begin{aligned} R_{HPS}(t) &= \left[1 - \left\{ \left(1 - e^{-\hat{\beta}_{AML}t} \right)^{\hat{\mu}_{AML}} \left(1 - e^{-\hat{\beta}_{BML}t} \right)^{\hat{\mu}_{BML}} \right\} \right] \\ & * \left[1 - \left\{ \left(1 - e^{-\hat{\beta}_{CML}t} \right)^{\hat{\mu}_{CML}} \left(1 - e^{-\hat{\beta}_{DML}t} \right)^{\hat{\mu}_{DML}} \right\} \right] \end{aligned} \quad (14)$$

Hybrid Series – Parallel System is a hybrid System consisting of (b) partial systems connected as parallel; within each partial System, there are (m). The reliability function for a hybrid series-parallel system (RHSP) can be calculated as follows:

$$RHSP(t) = 1 - \left[\prod_{k=1}^b \left\{ 1 - \prod_{j=1}^m R_{kj}(t) \right\} \right], j = 1,2, \dots, m, k = 1,2, \dots, b \quad (15)$$

For $k=4, j=2$, we have:

$$RHSP(t) = 1 - \left[\prod_{k=1}^2 \left\{ 1 - \prod_{j=1}^4 R_{kj}(t) \right\} \right] \tag{16}$$

$$RHSP(t) = 1 - \left[[1 - \{R_{1A}(t) * R_{1B}(t) * R_{1D}(t) * R_{1E}(t)\}] * [1 - \{R_{2A}(t) * R_{2B}(t) * R_{2D}(t) * R_{2E}(t)\}] \right] \tag{17}$$

For the components A, B, C and D, which followed the (GED) in Hybrid Parallel - Series System, the MLE of reliability say $\hat{R}HSP(t)$ is given as:

$$\hat{R}HSP(t) = 1 - \left[\left[1 - \left\{ \left(e^{-\hat{\beta}_{1A}t} \right)^{\hat{\mu}_{1A}} \left(e^{-\hat{\beta}_{1B}t} \right)^{\hat{\mu}_{1B}} \left(e^{-\hat{\beta}_{1C}t} \right)^{\hat{\mu}_{1C}} \left(e^{-\hat{\beta}_{1D}t} \right)^{\hat{\mu}_{1D}} \right\} \right] * \left[1 - \left\{ \left(e^{-\hat{\beta}_{2A}t} \right)^{\hat{\mu}_{2A}} \left(e^{-\hat{\beta}_{2B}t} \right)^{\hat{\mu}_{2B}} \left(e^{-\hat{\beta}_{2C}t} \right)^{\hat{\mu}_{2C}} \left(e^{-\hat{\beta}_{2D}t} \right)^{\hat{\mu}_{2D}} \right\} \right] \right] \tag{18}$$

3. Results and discussion

Since the (Deluxe) soap production process involves four stages (saponification and colour mixing, cutting and printing, packaging, and packaging), and since the linkage system for these machines is a series system with identical components, the General Company for the Vegetable Oil Industry was selected to collect data related to the research. Considering that the factory has two production lines, each of which has four stages, and that the average number of working hours per day in the factory is (8) hours, as shown in Table 1, the operating time of the four machines has been recorded in 1/2023. Statistical measurements of the operation time of the machines are shown in Table 2; we can see that the longest operation time is, and since the mean is greater than the median, all variables (machines) have a positive skewness and kurtosis.

Table 1. Some statistical measures

	m₁	m₂	m₃	m₄	m₅	m₆	m₇	m₈		m₁	m₂	m₃	m₄	m₅	m₆	m₇	m₈
1	4.4	3.8	4.2	4.0	4.3	3.7	4.1	3.9	12	6.1	5.3	5.8	5.6	6.0	5.2	5.7	5.5
2	3.7	3.2	3.5	3.4	3.6	3.2	3.5	3.3	13	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
3	1.7	1.5	1.6	1.5	1.6	1.4	1.6	1.5	14	2.0	1.7	1.9	1.8	1.9	1.7	1.8	1.8
4	0.9	0.8	0.8	0.8	0.9	0.8	0.8	0.8	15	1.0	0.9	1.0	1.0	1.0	0.9	1.0	0.9
5	1.6	1.4	1.5	1.4	1.5	1.3	1.4	1.4	16	0.6	0.6	0.6	0.6	0.6	0.5	0.6	0.6
6	0.8	0.7	0.8	0.8	0.8	0.7	0.8	0.7	17	1.8	1.5	1.7	1.6	1.7	1.5	1.6	1.6
7	0.9	0.8	0.9	0.8	0.9	0.8	0.9	0.8	18	2.5	2.2	2.4	2.3	2.4	2.1	2.3	2.2
8	1.1	1.0	1.1	1.0	1.1	1.0	1.1	1.0	19	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3
9	1.1	1.0	1.1	1.0	1.1	0.9	1.0	1.0	20	1.2	1.1	1.2	1.1	1.2	1.0	1.1	1.1
10	2.1	1.8	2.0	1.9	2.0	1.8	2.0	1.9	21	1.0	0.9	0.9	0.9	1.0	0.8	0.9	0.9
11	8.1	7.1	7.7	7.4	7.9	6.9	7.6	7.2	22	4.6	4.0	4.4	4.2	4.5	3.9	4.3	4.1

Table 2. Some statistical measures

Machine	Min	Max	Mean	Median	S.D.	Machine	Min	Max	Mean	Median	S.D.
m ₁	0.18	8.11	2.16	1.37	2.01	m ₅	0.17	7.92	2.11	1.34	1.96
m ₂	0.16	7.08	1.89	1.2	1.75	m ₆	0.15	6.39	1.84	1.17	1.72
m ₃	0.17	7.73	2.06	1.31	1.92	m ₇	0.17	7.56	2.01	1.28	1.87
m ₄	0.16	7.93	1.97	1.25	1.83	m ₈	0.16	7.23	1.93	1.22	1.79

The Kolmogorov Smirnov (K.S.) test at (0.05) significance level was conducted to know the distribution of the machines; the hypotheses were as follows:
H₀: The machine operation times follow GED.

Table 3. Kolmogorov Smirnov tests for operation time data

Machine	K-S	p-value	Machine	K-S	p-value
m₁	0.16071	0.5664	m ₅	0.16947	0.4995
m₂	0.21015	0.2485	m ₆	0.21793	0.2133
m₃	0.1784	0.4352	m ₇	0.18568	0.3864
m₄	0.19494	0.3293	m ₈	0.20248	0.2872

The p-values in Table 3 refer to accepting the above null hypotheses as greater than the significance level. Thus, the operation data of the four machines are followed (GED). Also, the box plots in Figure 1 represent the operating times for these machines; it is clear from it that these data suffer from outliers.

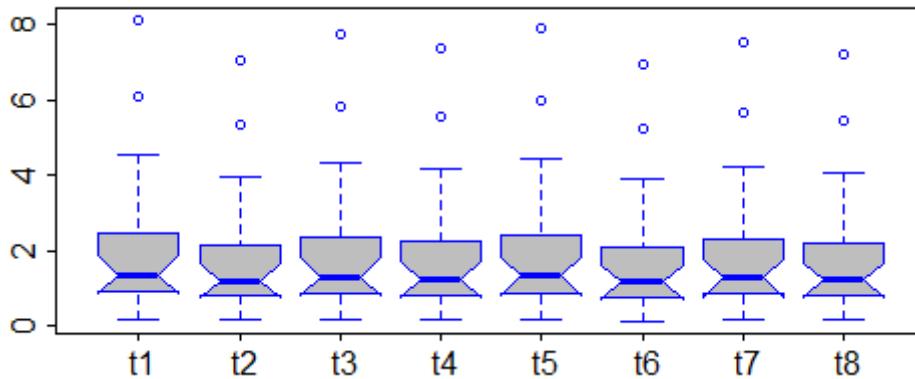


Figure 1. Box plots for data

Table 4 depicts the estimated values of (μ) and (β) for the machines by using the (ML) methods, also the (S.E.) values for these estimators; note that the (S.E.) values for estimating (μ) are smaller than the (S.E.) values for estimating (β).

The reliability functions have been estimated using the ML method for all eight machines, the factory work system, i.e. the hybrid series-parallel System, and the employed hybrid System, which is the series-parallel hybrid System, the results shown in Table 5 and Figure 2.

Table 4. Estimated parameters values

Machine	Parameter	Estimate	S.E.	Machine	Parameter	Estimate	S.E.
m₁	$\hat{\mu}$	1.4459	0.3802	m ₅	$\hat{\mu}$	1.3827	0.3264
	$\hat{\beta}$	0.5516	0.1208		$\hat{\beta}$	0.538	0.1006
m₂	$\hat{\mu}$	1.2677	0.2283	m ₆	$\hat{\mu}$	1.2124	0.1811
	$\hat{\beta}$	0.5379	0.0579		$\hat{\beta}$	0.5223	0.036
m₃	$\hat{\mu}$	1.3772	0.3217	m ₇	$\hat{\mu}$	1.3468	0.2958
	$\hat{\beta}$	0.5453	0.0984		$\hat{\beta}$	0.5438	0.0885
m₄	$\hat{\mu}$	1.2939	0.2507	m ₈	$\hat{\mu}$	1.2868	0.2446
	$\hat{\beta}$	0.5312	0.0692		$\hat{\beta}$	0.5371	0.0657

Table 5. Estimated reliability functions

t	\hat{R}_1	\hat{R}_2	\hat{R}_3	\hat{R}_4	\hat{R}_5	\hat{R}_6	\hat{R}_7	\hat{R}_8	\hat{R}_{HSP}	\hat{R}_{HPS}
1	0.6637	0.6250	0.6505	0.6376	0.6571	0.6188	0.6440	0.6313	0.3952	0.9809
2	0.4404	0.3906	0.4232	0.4066	0.4317	0.3829	0.4148	0.3985	0.0637	0.7924
3	0.2923	0.2441	0.2753	0.2592	0.2837	0.2369	0.2671	0.2516	0.0084	0.4779
4	0.1940	0.1526	0.1791	0.1653	0.1864	0.1466	0.1720	0.1588	0.0010	0.2304
5	0.1287	0.0954	0.1165	0.1054	0.1225	0.0907	0.1108	0.1003	0.0001	0.0965

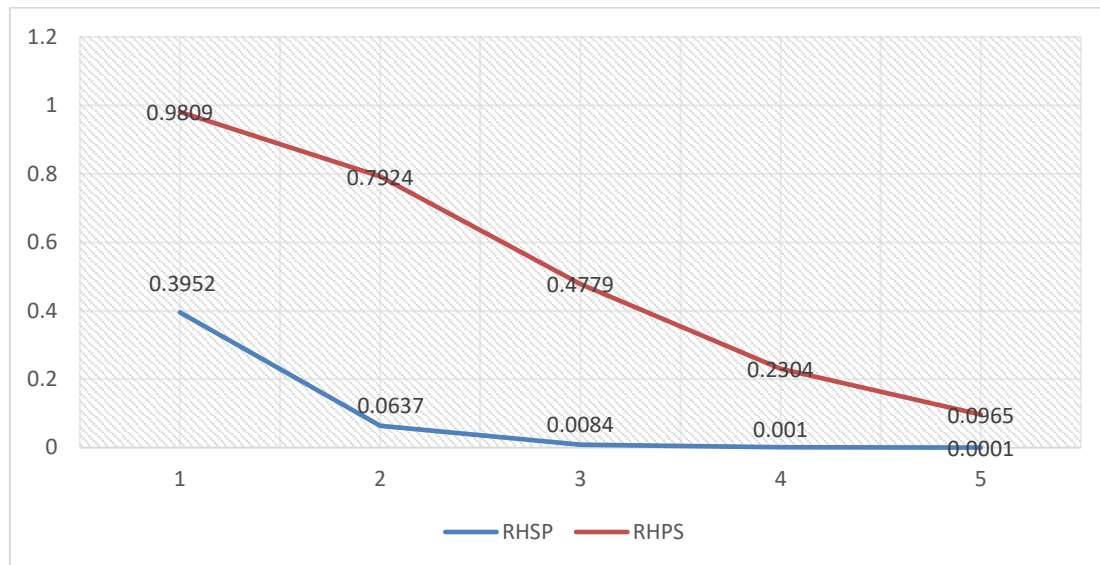


Figure 2. Estimated reliability functions

4. Conclusions

Through the values of the reliability functions for the eight machines, it is clear that the eight machines can work for one hour without stopping with a probability of more than 60%, while the probability that those machines will stop in the second hour is more than 50%. Thus the probability that the factory will be out of work during the first or second-hour System is very high; this indicates a clear problem with the factory working system. But suppose the employed hybrid System, i.e. the hybrid parallel-series system, is applied. In that case, the probability that the factory is out of work during the first hour will fade away. The probability that the factory will not stop during the second hour becomes 80%, and the probability that the factory will not stop during the third hour becomes 47%.

Declaration of competing interest

The authors declare that they have no known financial or non-financial competing interests in any material discussed in this paper.

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References

- [1] H. M. Almongy and E. M. Almetwally, "Robust estimation methods of generalized exponential distribution with outliers," *Pak. J. Stat. Oper. Res.*, pp. 545–559, 2020.
- [2] M. Al-Nidawi, "Comparison Some Estimators of reliability function for hybrid asymmetry system (Parallel/Serial) with Practical Application," Al-Mustansiriyah University, Baghdad, Iraq, 2017.
- [3] A. Al-Saady, "Modeling and Estimating the reliability function of some systems in the case of asymmetry of their compounds," Al-Mustansiriyah University, 2016.
- [4] H. S. Migon, D. Gamerman, and F. Louzada, *Statistical inference: An integrated approach, second edition*, 2nd ed. Caithness, U.K.: Whittles Publishing, 2014.
- [5] R. Gupta and D. Kundu, "Generalized exponential distributions: different methods of estimation," *Journal of Statistical Computer and Simulation*, vol. 69, pp. 315–338, 2001.
- [6] M. H. Tahir, G. M. Cordeiro, M. Alizadeh, M. Mansoor, M. Zubair, and G. G. Hamedani, "The odd generalized exponential family of distributions with applications," *J. Stat. Distrib. Appl.*, vol. 2, no. 1, 2015.

- [7] R. D. Gupta and D. Kundu, "Generalized exponential distribution: Existing results and some recent developments," *J. Stat. Plan. Inference*, vol. 137, no. 11, pp. 3537–3547, 2007.
- [8] B. Pradhan and D. Kundu, "On progressively censored generalized exponential distribution," *Test (Madr.)*, vol. 18, no. 3, pp. 497–515, 2009.
- [9] M. Z. Raqab, "Inferences for generalized exponential distribution based on record statistics," *J. Stat. Plan. Inference*, vol. 104, no. 2, pp. 339–350, 2002.
- [10] T. P. Ryan and W. Q. Meeker, "System reliability theory: Models, statistical methods, and applications, second edition," *J. Qual. Technol.*, vol. 37, no. 1, pp. 84–87, 2005.