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The random coefficient autoregressive model with seasonal volatility innovations (RCA-SGARCH)

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ABSTRACT

This paper dealt with the autoregressive model when the coefficient is random. The residuals series of the model exhibit two behaviors, kurtosis, and volatility. These volatilities are usually seasonal in the real financial data, which always uses GARCH models. So, the use of RCA and GARCH models together will provide an appropriate framework to study and analysis of time-varying volatility as well as the presence of seasonal effects in financial series. Applying copper's daily economic close prices when the errors series are distributed, as usual, $t_{(3)}$ and $t_{(7)}$ distributions are achieved. Therefore, the RCA(1) model, when residuals follow the GARCH(1, 0)x(0, 1)$_5$ model together, is the appropriate model.

Keywords: RCA(1), GARCH, volatility, kurtosis.

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1. Introduction

The volatility can generally be defined as deviations and unexpected movements that appear in time series, especially financial ones. Therefore, most financial economists are concerned with return volatilities as they measure risk.

Other studies have shown that the pattern of much financial time series, such as stock returns, exhibit two behaviors, leptokurtosis, and time-varying volatility (Engle, [8]; Nicholls & Quinn, [13]; Bollerslev, [3]). Thus, the appropriate model for formulating the behaviors together is the generalized autoregressive conditional heteroscedasticity (GARCH) and the random coefficient autoregressive (RCA) models. Thavaneswaran et al. [15] derive the kurtosis of various GARCH models such as non-Gaussian GARCH, nonstationary, and random coefficient GARCH. Thavaneswaran et al. [14] derive the general properties for RCA models with GARCH innovations such as mean, variance, and kurtosis under autoregressive assumptions. Frank et al. [10] derive the kurtosis of RCA with seasonal GARCH, the variance of the l-steps ahead forecast errors, and the kurtosis of the error distribution. Gorka [6] found out that the RCA-GARCH models can be successfully used for pricing options for the dynamics of the volatility. Goryainov and Goryainova [11] proved the asymptotic normality of the minor absolute deviations estimate for the autoregressive models with a random coefficient. In much financial time series, such as foreign exchange rates, volatility or seasonal volatility effects and conditional non-normality can induce the leptokurtosis typically observed in economic data. These features are not appropriate for well-known models when used in future forecasting. So, other proper models must be used to study volatility or its seasonal effects in financial markets. Accordingly, RCA models have been used with the seasonal GARCH model as they provide this framework.

Our research is designed to study and analyze the Random coefficient Autoregressive Model with seasonal volatility innovations (RCA-SGARCH) and using real data of daily world copper price series (minimum prices) to estimate the appropriate studied model and then compare it with the same process when the errors series distributed as *T- distribution* with degree of freedom (3) and (5).

2. RCA models

The Random coefficient autoregressive time series were introduced by Nicholls and Quinn [13], and some of their properties have been studied recently by Appadoo et al. [2]. RCA models exhibiting extended memory properties have been considered in Leipus and Surgallis [12]. A sequence of random variables $\{y_t\}$ is called an RCA(1) time series if it satisfies the equations:

$$y_t = (\beta + b_t)\, y_{t-1} + e_t \tag{1}$$

t ∈ Z, where Z denotes the set of integers, β is a real parameter and

(i) $\begin{pmatrix} b_t \\ e_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_b^2 & 0 \\ 0 & \sigma_e^2 \end{pmatrix}\right)$

(ii) $\beta^2 + \sigma_b^2 < 1$.

Where $\{b_t\}$ and $\{e_t\}$ are the errors sequences in the model. According to Nicholls and Quinn [13], condition (ii) is necessary and sufficient for the second-order stationarity of $\{y_t\}$.
The RCA model parameters are usually estimated using the least squares method [1,4].

Theorem 1. Let $\{y_t\}$ be an RCA(1) time series satisfying conditions (i) and (ii), and let γ_y be its covariance function. Then,

(a) $Ey_t = 0$, $Ey_t^2 = \frac{\sigma_e^2}{1-\beta^2-\sigma_b^2}$, the kth lag autocovariance for y_t is given by $\gamma_y(k) = \frac{\beta^k \sigma_b^2}{1-\beta^2-\sigma_b^2}$ and the autocorrelation for y_t is $\rho_k = \beta^k$ for all k ∈ Z.

(b) If $\{b_t\}$ and $\{e_t\}$ are normally distributed random variables and if e_t and b_t are correlated with correlation coefficient ρ, then the kurtosis $K^{(y)}$of the RCA process $\{y_t\}$ is given by

$$K(y) = \frac{6(\sigma_b^2 + \beta^2)[1 - \beta^3 - 3\beta\sigma_b^2] + 72\beta^3\rho^2\sigma_b^2 + 3[1 - (\beta^2 + \sigma_b^2)][1 - \beta^3 - 3\beta\sigma_b^2]}{[1 - \beta^3 - 3\beta\sigma_b^2][1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]}$$

$$*[1 - (\beta^2 + \sigma_b^2)]\,. \tag{2}$$

On the other hand, if e_t and b_t are uncorrelated, then $\rho = 0$, and the kurtosis $K^{(y)}$of the RCA process $\{y_t\}$ is given by:

$$K^{(y)} = \frac{3\left[1 - (\beta^2 + \sigma_b^2)^2\right]}{\left[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4\right]} \tag{3}$$

And if e_t and b_t are correlated with correlation coefficient ρ, then the skewness $S^{(y)}$ of the RCA process $\{y_t\}$ is given by :

$$S^{(y)} = \frac{6\beta\rho\sigma_b \left[1 - (\beta^2 + \sigma_b^2)\right]^{1/2}}{[1 - \beta^3 - 3\beta\,\sigma_b^2]} \tag{4}$$

If e_t and b_t are uncorrelated, then $\rho = 0$, and the skewness $S^{(y)} = 0$.

3. GARCH model

The general class of these models which introduced by Bollerslev (1986) [3] can be written as:

$$e_t = \sqrt{h_t}\, Z_t \tag{5}$$
$$h_t = \omega + \sum_{i=1}^{p} \alpha_i\, e_{t-i}^2 + \sum_{j=1}^{q} \beta_j\, h_{t-j}, \tag{6}$$

Where h_t is the conditional variance of the returns, Z_t is (i.i.d.) random variables with $EZ_t = 0$ and $Var(Z_t) = 0$. Let $u_t = e_t^2 - h_t$ and σ_u^2 is the variance of u_t. Then the model could be as:

$$e_t^2 - u_t = \omega + \sum_{i=1}^{p} \alpha_i e_{t-i}^2 + \sum_{j=1}^{q} \beta_j \left(e_{t-j}^2 - u_{t-j}\right)$$

$$\left[1 - \sum_{i=1}^{p} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j\right] e_t^2 = \omega - \sum_{j=1}^{q} \beta_j u_{t-j}$$

$$\phi(B) e_t^2 = \omega + \beta(B) u_t, \tag{7}$$

Where $\phi(B) = \sum_{i=1}^{r} \phi_i B^i$, $\phi_i = (\alpha_i + \beta_i)$, $\beta(B) = \sum_{j=1}^{q} \beta_j$.

4. Multiplicative seasonal GARCH models

This model can be written as follows [7]:

$$e_t = \sqrt{h_t} Z_t \tag{8}$$

$$\theta(B)\Theta(L) h_t = \omega + \alpha(B) e_t^2 \tag{9}$$

Where $\{Z_t\}$ is (i.i.d.) random variables with $EZ_t = 0$ and $Var(Z_t) = 0$.

$$\alpha(B) = \theta(B)\Theta(L) - \phi(B)\Phi(L) ,$$

$$\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i , \theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i$$

$$\Phi(L) = 1 - \sum_{i=1}^{P} \Phi_i L^i , \Theta(L) = 1 - \sum_{i=1}^{Q} \theta_i L^i \text{ , and } L = B^s$$

Letting $u_t = e_t^2 - h_t$, then equation (9) may be written as a seasonal $ARMA\ (p, q)x(P, Q)s$ of ϵ_t^2.

$$\theta(B)\Theta(L)[e_t^2 - u_t] = \omega + [\theta(B)\Theta(L) - \phi(B)\Phi(L)] e_t^2$$

Then,

$$\phi(B)\Phi(L)\, e_t^2 = \omega + \theta(B)\Theta(L) u_t \tag{10}$$

5. RCA - SGARCH Model

The RCA - SGARCH model has been suggested by (Frank, Ghahramani & Thavaneswarant, 2011 [11]). Then, by using the same transformation when obtaining equation (10), the general form of the model is given by :

$$y_t = (\beta + b_t)\, y_{t-1} + e_t \tag{11}$$

$$e_t = \sqrt{h_t} Z_t \tag{12}$$

$$\phi(B)\Phi(L)\, e_t^2 = \omega + \theta(B)\Theta(L) u_t \tag{13}$$

Where $Z_t, \phi(B), \Phi(L), \theta(B),$ and $\Theta(L)$ were defined in section 6, and e_t^2, as given in (10), is stationary.

Consider the RCA(1) $-$ SGARCH $(1, 0)x(0, 1)$ process:

$$y_t = (\beta + b_t)\, y_{t-1} + e_t \tag{14}$$

$$e_t = \sqrt{h_t}\, Z_t \tag{15}$$

$$(1-\phi B)e_t^2 = \omega + (1-\Theta L)u_t \tag{16}$$

where $u_t = e_t^2 - h_t$, and ψ -weights are given by:

$\psi_1 = \phi$,..., $\psi_{s-1} = \phi^{s-1}$, $\psi_s = (\phi^s - \Theta)$, $\psi_{s+j} = \phi^j \psi_s$, $j \geq 1$. It can be shown that $\sum_{j=0}^{\infty} \psi_j^2 = \frac{[1+(\phi^s - \Theta)^2]}{1-\phi^2}$. Then we have the following expectations of to find the kurtosis of the process.

$$E(y_t^2) = \frac{E(e_t^2)}{[1-\beta^2-\sigma_b^2]}$$

$$E(y_t^4) = \frac{6\,[\sigma_b^2+\beta^2]\{E(e_t^2)\}^2}{[1-(\beta^2+\sigma_b^2)][1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{E(e_t^4)}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}$$

$$k^{(y)} = \frac{6\,[\sigma_b^2+\beta^2][1-(\beta^2+\sigma_b^2)]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{[1-(\beta^2+\sigma_b^2)]^2}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}\; k^{(e)} \tag{17}$$

If Z_t is distributed as normal and b_t is noise process and correlated $Eb_t = 0$ and $Var(b_t) = \sigma_b^2$. Then,

$$E(y_t^2) = \frac{E(h_t)}{[1-\beta^2-\sigma_b^2]}$$

$$E(y_t^4) = \frac{6\,[\sigma_b^2+\beta^2]\{E(h_t)\}^2}{[1-(\beta^2+\sigma_b^2)][1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{3E(h_t^2)}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}$$

$$k^{(y)} = \frac{6\,[\sigma_b^2+\beta^2][1-(\beta^2+\sigma_b^2)]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{3[1-(\beta^2+\sigma_b^2)]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}\;\frac{E(h_t^2)}{\{E(h_t)\}^2}$$

where**,**

$$\frac{E(h_t^2)}{\{E(h_t)\}^2} = \frac{1}{E(Z_t^4) - \{E(Z_t^4)-1\}\sum_{j=0}^{\infty}\psi_j^2} \tag{18}$$

For a standard normal variate $E(Z_t^4) = 3$, then the kurtosis of y_t is:

$$k^{(y)} = \frac{6\,[\sigma_b^2+\beta^2][1-(\beta^2+\sigma_b^2)]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{3[1-(\beta^2+\sigma_b^2)]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]\{3-2\frac{[1+(\phi^s-\Theta)^2]}{1-\phi^2}\}} \tag{19}$$

The forecast error variance of the series y_{n+1} is denoted by $e_n^{(y)}(l)$, where (l) is the steps-ahead, and the variance of $e_n^{(y)}(l)$ for the $RCA(1) - GARCH(1,0)x(0,1)_s$ model is given by:

$$var\left[e_n^{(y)}(l)\right] = \frac{\omega(1-\beta^2)}{(1-\phi\,)[1-(\beta^2+\sigma_b^2)]}\sum_{i=1}^{l-1} B^{2j} \tag{20}$$

6. Estimating RCA (1)

Let $y_n, y_n, \dots, y_n$ be a sample size n observations, and the RCA(1) process in (1) may be written as follows [1]:

$$y_t = \beta\, y_{t-1} + v_t \tag{21}$$

Where $v_t = b_t\, y_{t-1} + e_t$,

then the least squares estimate $\hat{\beta}_{ls}$ of β is given by:

$$\hat{\beta}_{ls} = \frac{\sum_{t=2}^{n} y_t y_{t-1}}{\sum_{t=2}^{n} y_{t-1}^2} \tag{22}$$

After estimating β, we may then estimate σ_b^2 for all i and σ_e^2 by considering equation (21) such that

$$\hat{v}_{t,ls} = y_t - \hat{\beta}_{ls} y_{t-1} \tag{23}$$

$$\sigma_{b,ls}^2 = \frac{\sum_{t=2}^{n} \hat{v}_{t,ls}^2 (y_{t-1}^2 - \bar{z})}{\sum_{t=2}^{n} y_{t-1}^4 - \bar{z}^2} \tag{24}$$

$$\sigma_{e,ls}^2 = \frac{\sum_{t=2}^{n} \hat{v}_{t,ls}^2 - \sigma_{b,ls}^2 \bar{z}}{n-1} \tag{25}$$

Where $\bar{z} = \sum_{t=2}^{n} \frac{y_{t-1}^2}{n-1}$

7. Application

The sample representing the time series of copper's daily financial close prices in US dollars per pound from 10/1/2010 to 29/5/2015 was approved. Figure 1 depicts the graph of the studied time series observations. It is illustrated by a graph that there is a trend with various volatilities, and then the series is nonstationary in the mean. An Augmented Dicky – Fuller test was performed, with a value (DF=-2.4579) and probability of (P-value=0.1262), indicating that the series was nonstationary. Accordingly, the first difference was taken as in Figure 2, which confirms that the series becomes stationary.

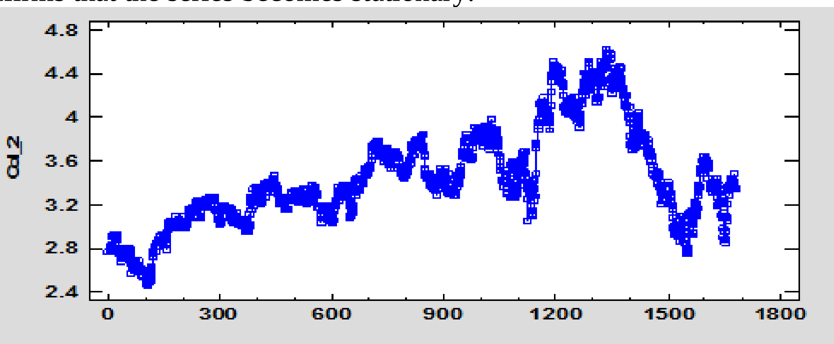

Figure 1. Daily closing prices of copper time series

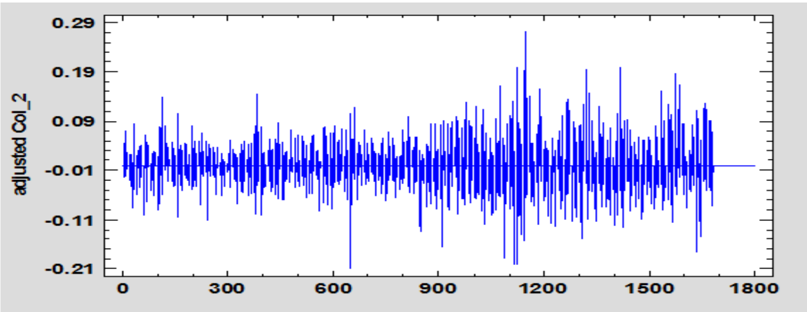

Figure 2. Daily closing prices of copper time series with the first difference

After preparing the series, the random coefficient autoregressive process RCA by using the traditional least squares (LS) was estimated as follows:

Table 1. Results of parameters estimation of RCA (1) model

statistics	Estimate	Standard Error	Probability
$\hat{\beta}_{ls}$	0.99321	0.10874	0.0001
$\hat{\sigma}_b^2$	0.00152	0.00035	0.0003
$\hat{\sigma}_e^2$	0.00231	0.00051	0.0000

Hence, employing a series of residuals resulting from the estimation of the RCA process to build the GARCH process. It shows that the e_t series is leptokurtic ($kur = 5.664$) and that the volatilities in the series are apparent. Therefore, the GARCH model should be applied to the residuals series of 1682 observations plotted in Figure 3.

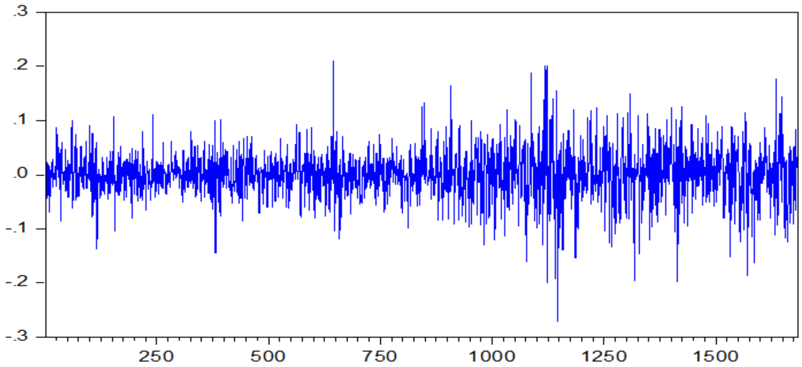

Figure 3. Residuals series of Copper data

Figures 4 and 5 show graphs of the coefficients of the (ACF) and (PACF) of the squared residuals. The ACF and PACF feature exponential decay with significant points at seasonal lags with seasonality index s = 5. To investigate the existence of the ARCH effect for the residuals of the RCA model, the Lagrange multiplier test was used, where the value of (LM-test=891.26) with probability (P-value = 0.000), so the null hypothesis stating that there is no effect of ARCH was rejected. Thus, the residuals series has an ARCH effect.

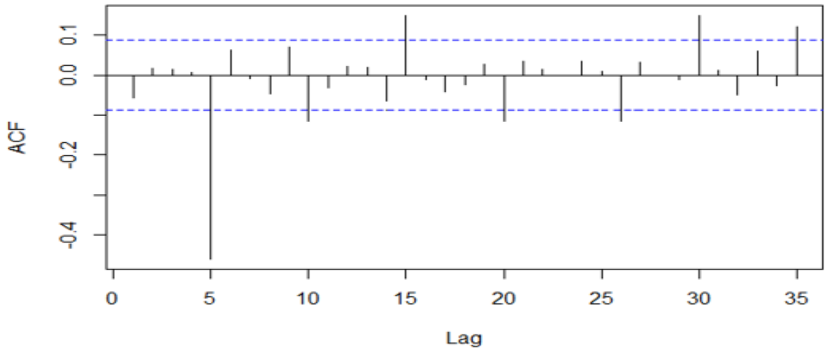

Figure 4. The ACF coefficients of squared residuals (Series Yt)

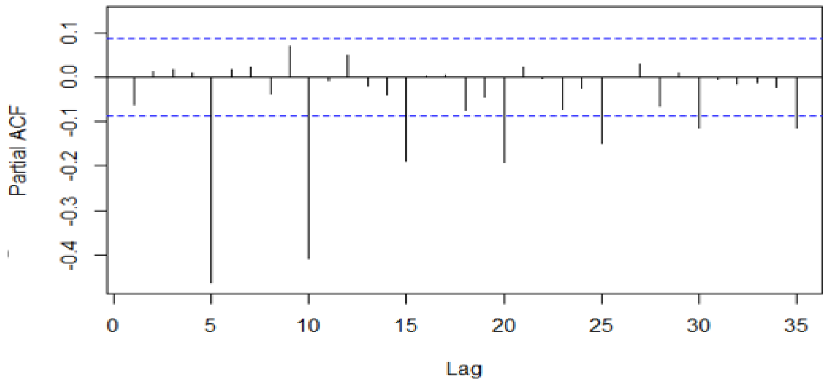

Figure 5. The PACF coefficients of squared residuals (Series Yt)

The residual series was also stationary since (DF=- 43.47482) and the probability of (P-value=0.0001). A seasonal GARCH(1,0)x$(0,1)_5$ model given by (15)-(16) with standard normal Z_t was estimated by (MLE) for the copper data, and the results of the estimation are shown in Table 2. After evaluating a set of a running model by the maximum likelihood method when (s=5), it turns out that GARCH(1,0)x$(0,1)_5$ is the appropriate model.

Table 2. The results of estimation of the GARCH(1,0)x$(0,1)_5$ model

Parameters	Parameters Est.	Stand. Error of Estimator	Probability
μ	0.000277	0.000072	0.0002
ω	5.74E-07	2.97E-08	0.0000
ϕ	0.1283	0.00609	0.0000
Θ	0.8807	0.00487	0.0000

Finally, the Lagrange multiplier test is applied to verify the estimated model's suitability. Table 3 found that the probability of the test (P-value) was the largest (0.05), which refers to the lack of effect ARCH.

Table 3. ARCH test of the estimated model

Test	Value of test	Probability
F-statistic	0.053931	0.8164
LM- test	0.053994	0.8163

Then, the seasonal GARCH $(1, 0)x(0, 1)_5$ model for the residuals is given by:

$$h_t = \omega + \phi e_{t-1}^2 - \Theta e_{t-5}^2 + \Theta h_{t-5}$$
$$h_t = 5.74\text{E} - 07 + 0.1283 e_{t-1}^2 - 0.8807 e_{t-5}^2 + 0.8807 h_{t-5}$$

And for comparison, A model GARCH$(1, 0)x(0, 1)_5$ model was fitted individually for the origin data by (MLE) using two distributions of residuals, $t_{(3)}$ and $t_{(7)}$. And the results of the estimation are shown in Table (2) as follows

Table 4. The results of estimation of the GARCH(1,0)x$(0,1)_5$ model for the origin data

Parameters	Normal(0,1) errors			$t_{(3)}$ errors		$t_{(7)}$ errors	
	Parameters Est.	Stand .E. of estimators		Parameters Est.	Stand .E. of estimators	parameters Est.	Stand .E. of estimators
μ	.00027	.00007		.00043	.00021	.00068	.00039
ω	5.74E- 7	2.97E-8		2.77E-5	1.39E-5	1.65E-5	7.01E-6
ϕ	0.1283	0.00609		0.06100	0.01603	0.03929	0.00816
Θ	0.8807	0.00487		0.95383	0.01166	0.95389	0.00935
AIC	-8.0483			-3.4079		-3.4216	

8. Conclusion

In this work, the RCA-SGARCH process is considered. The properties of moments and kurtosis of the volatility model are presented. The model studied in the paper is an example of the seasonal volatility model. High moments and kurtosis properties are also shown under the assumption of the normal distribution when (e_t) and (b_t) are correlated and not. An application of the daily financial close prices of copper is used. Then we made a comparison among the studied model with errors distributed as $N(0,1)$ and the same process when the errors series were distributed as *T- distribution* with degrees of freedom (3) and (5). It shows that the studied model is the best according to its parameters' significance and the Akaike information criterion (AIC) value. Also, to compare the process when the errors series are distributed as $t_{(7)}$ and $t_{(3)}$, the model with $t_{(7)}$ is the better since the standard error of the parameters and AIC of it is the least.

Declaration of competing interest

There are no financial or non-financial competing interests in this paper's content, according to its authors.

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APPENDIX

Proof for Theorems 1

$$y_t = (\beta + b_t)\, y_{t-1} + e_t$$

$$y_t^2 = \beta^2 y_{t-1}^2 + b_t^2 y_{t-1}^2 + 2\beta b_t y_{t-1}^2 + 2\beta e_t y_{t-1} + 2b_t e_t y_{t-1} + e_t^2$$

$$E(y_t^2) = \beta^2 E(y_{t-1}^2) + E(b_t^2 y_{t-1}^2) + 2\beta E(b_t y_{t-1}^2) + 2\beta E(e_t y_{t-1}) + 2E(b_t e_t y_{t-1}) + E(e_t^2)$$
$$= \beta^2 E(y_{t-1}^2) + \sigma_b^2 E(y_{t-1}^2) + E(e_t^2)$$

$$E(y_t^2) = \frac{\sigma_e^2}{[1 - \beta^2 - \sigma_b^2]}$$

$$y_t^3 = \beta^3 y_{t-1}^3 + 3\beta^2 y_{t-1}^3 b_t + 3\beta^2 y_{t-1}^2 e_t + 3\beta y_{t-1}^3 b_t^2 + 6\beta\, y_{t-1}^2 b_t e_t + 3\beta y_{t-1} e_t^2 + b_t^3 y_{t-1}^3 + 3b_t^2 y_{t-1}^2 e_t + 3b_t y_{t-1} e_t^2 + e_t^3$$

$$E(y_t^3) = \beta^2 E(y_{t-1}^3) + 3\beta\sigma_b^2 E(y_{t-1}^3) + 6\beta\,\rho\sigma_b\,\sigma_e E(y_{t-1}^2)$$

$$E(y_t^3) = \beta^2 E(y_{t-1}^3) + 3\beta\sigma_b^2 E(y_{t-1}^3) + 6\beta\,\rho\sigma_b\,\sigma_e \frac{\sigma_e^2}{[1 - \beta^2 - \sigma_b^2]}$$

$$E(y_t^3) = \beta^2 E(y_{t-1}^3) + 3\beta\sigma_b^2 E(y_{t-1}^3) + \frac{6\beta\,\rho\sigma_b\sigma_e^3}{[1 - \beta^2 - \sigma_b^2]}$$

$$E(y_t^3) = \frac{6\beta\,\rho\sigma_b\sigma_e^3}{[1 - \beta^2 - 3\beta\sigma_b^2][1 - (\beta^2 + \sigma_b^2)]}$$

Then, the skewness is given by:

$$S^{(y)} = \frac{E(y_t^3)}{[E(y_t^2)]^{3/2}}$$
$$= \frac{\dfrac{6\beta\rho\sigma_b\sigma_e^3}{[1 - \beta^2 - 3\beta\sigma_b^2][1 - (\beta^2 + \sigma_b^2)]}}{\left[\dfrac{\sigma_e^2}{[1 - \beta^2 - \sigma_b^2]}\right]^{3/2}}$$
$$= \frac{6\beta\,\rho\sigma_b\sigma_e^3}{[1 - \beta^2 - 3\beta\sigma_b^2][1 - (\beta^2 + \sigma_b^2)]} \cdot \frac{[1 - (\beta^2 + \sigma_b^2)]^{3/2}}{\sigma_e^3}$$

$$S^{(y)} = \frac{6\beta\,\rho\sigma_b[1 - (\beta^2 + \sigma_b^2)]^{1/2}}{[1 - \beta^2 - 3\beta\sigma_b^2]}$$

If e_t and b_t are uncorrelated then $\rho = 0$,then $S^{(y)} = 0$

$$E(y_t^4) = 6\beta^2 E(y_{t-1}^4 b_t^2) + 6E(b_t^2 y_{t-1}^2 e_t^2) + \beta^4 E(y_{t-1}^4) + E(b_t^4 y_{t-1}^4) + E(e_t^4) + 12\beta^2 E(y_{t-1}^3 b_t e_t) + 6\beta^2 E(y_{t-1}^2 e_t^2)$$
$$= 6\beta^2\sigma_b^2 E(y_{t-1}^4) + 6\sigma_b^2\sigma_e^2 E(y_{t-1}^2) + \beta^4 E(y_{t-1}^4) + 3\sigma_b^4 E(y_{t-1}^4) + 3\sigma_e^4 + 12\beta^2\rho\sigma_b\sigma_e E(y_{t-1}^3) + 6\beta^2\sigma_e^2 E(y_{t-1}^2)$$

$$[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]E(y_t^4) = 6\sigma_e^2 E(y_{t-1}^2)[\sigma_b^2 + \beta^2] + 12\beta^2\rho\sigma_b\sigma_e E(y_{t-1}^3) + 3\sigma_e^4$$

$$[1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]E(y_t^4) = \frac{6\sigma_e^4\,[\sigma_b^2 + \beta^2]}{[1 - (\beta^2 + \sigma_b^2)]} + \frac{72\beta^3\rho^2\sigma_b^2\sigma_e^4}{[1 - (\beta^2 + \sigma_b^2)][1 - \beta^2 - 3\beta\sigma_b^2]} + 3\sigma_e^4$$

$$E(y_t^4) = \left\{\frac{6\sigma_e^4\,[\sigma_b^2 + \beta^2]}{[1 - (\beta^2 + \sigma_b^2)]} + \frac{72\beta^3\rho^2\sigma_b^2\sigma_e^4}{[1 - (\beta^2 + \sigma_b^2)][1 - \beta^2 - 3\beta\sigma_b^2]} + 3\sigma_e^4\right\} / [1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]$$

$$k^{(y)} = \frac{6(\sigma_b^2 + \beta^2)[1 - \beta^3 - 3\beta\sigma_b^2] + 72\beta^3\rho^2\sigma_b^2 + 3[1 - (\beta^2 + \sigma_b^2)][1 - \beta^3 - 3\beta\sigma_b^2]}{[1 - \beta^3 - 3\beta\sigma_b^2][1 - 6\beta^2\sigma_b^2 - \beta^4 - 3\sigma_b^4]} * [1 - (\beta^2 + \sigma_b^2)]\,.$$

If e_t and b_t are uncorrelated then $\rho = 0$, then:

$$E(y_t^4) = \frac{3\sigma_e^4[1+(\beta^2+\sigma_b^2)]}{[1-(\beta^2+\sigma_b^2)][1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}$$

$$k^{(y)} = \frac{3\left[1-(\beta^2+\sigma_b^2)^2\right]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}$$

Proof for equation (17)

$$y_t^2 = \beta^2 y_{t-1}^2 + b_t^2 y_{t-1}^2 + 2\beta b_t y_{t-1}^2 + 2\beta e_t y_{t-1} + 2b_t e_t y_{t-1} + e_t^2$$

$$E(y_t^2) = \beta^2 E(y_{t-1}^2) + E(b_t^2 y_{t-1}^2) + 2\beta E(b_t y_{t-1}^2) + 2\beta E(e_t y_{t-1}) + 2E(b_t e_t y_{t-1}) + E(e_t^2)$$
$$= \beta^2 E(y_{t-1}^2) + \sigma_b^2 E(y_{t-1}^2) + E(e_t^2)$$

$$E(y_t^2) = \frac{E(e_t^2)}{[1-\beta^2-\sigma_b^2]}$$

$$E(y_t^4) = 6\beta^2 E(y_{t-1}^4 b_t^2) + 6E(b_t^2 y_{t-1}^2 e_t^2) + \beta^4 E(y_{t-1}^4) + E(b_t^4 y_{t-1}^4) + E(e_t^4) + 6\beta^2 E(y_{t-1}^2 e_t^2)$$
$$= 6\beta^2\sigma_b^2 E(y_{t-1}^4) + 6\sigma_b^2 E(y_{t-1}^2)E(e_t^2) + \beta^4 E(y_{t-1}^4) + 3\sigma_b^4 E(y_{t-1}^4) + E(e_t^4) + 6\beta^2 E(y_{t-1}^2)E(e_t^2)$$

$$[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]E(y_t^4) = 6E(e_t^2)[\sigma_b^2+\beta^2]E(y_{t-1}^2) + E(e_t^4)$$

$$[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]E(y_t^4) = \frac{6\ [\sigma_b^2+\beta^2]\{E(e_t^2)\}^2}{[1-(\beta^2+\sigma_b^2)]} + E(e_t^4)$$

$$E(y_t^4) = \frac{6\ [\sigma_b^2+\beta^2]\{E(e_t^2)\}^2}{[1-(\beta^2+\sigma_b^2)][1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{E(e_t^4)}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}$$

$$k^{(y)} = \frac{6\ [\sigma_b^2+\beta^2][1-(\beta^2+\sigma_b^2)]}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]} + \frac{[1-(\beta^2+\sigma_b^2)]^2}{[1-6\beta^2\sigma_b^2-\beta^4-3\sigma_b^4]}\ k^{(e)}$$

Proof for equation (18)

From equation (7), $\phi(B)e_t^2 = \omega + \beta(B)u_t$ where $u_t = e_t^2 - h_t$. For any stationary process,

$$var(e_t^2) = \sigma_u^2[\psi_1^2 + \psi_2^2 + \cdots],$$
$$\sigma_u^2 = E(e_t^4) - E(h_t^2)$$
$$= E(h_t^2 Z_t^4) - E(h_t^2) = E(h_t^2)E(Z_t^4) - E(h_t^2)$$
$$= E(h_t^2)[E(Z_t^4) - 1]$$

$$\therefore\ var(e_t^2) = E(h_t^2)[E(Z_t^4)-1][\psi_1^2+\psi_2^2+\cdots]. \qquad \text{(A.1)}$$

But from equation (7), it follows that,

$$var(e_t^2) = E(e_t^4) - [E(e_t^2)]^2$$
$$= E(h_t^2 Z_t^4) - [E(h_t)]^2$$
$$= E(h_t^2)E(Z_t^4) - [E(h_t)]^2 \qquad \text{(A.2)}$$

Equating (A.1) and (A.2),

$$E(h_t^2)E(Z_t^4) - [E(h_t)]^2 = E(h_t^2)[E(Z_t^4)-1][\psi_1^2+\psi_2^2+\cdots]$$

$$E(Z_t^4) - \frac{[E(h_t)]^2}{E(h_t^2)} = [E(Z_t^4)-1][\psi_1^2+\psi_2^2+\cdots]$$

$$\frac{[E(h_t)]^2}{E(h_t^2)} = E(Z_t^4) - [E(Z_t^4)-1][\psi_1^2+\psi_2^2+\cdots]$$