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Coefficient Estimates and Fekete- Szegő Inequality for a Subclass of Bi-Univalent Functions Defined by Symmetric Q-Derivative Operator by Using Faber Polynomial Techniques

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ABSTRACT

In this article we have defined a subclass of Bi-univalent functions using symmetric q- derivative operator and estimated the bounds for the coefficients using Faber polynomial techniques. We also have obtained the bounds for the linear functional which is popularly known as Fekete- Szegő problem.

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1. Introduction

Let A be the class of all normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Which are analytic in the Unit disk U.
A function that is regular (holomorphic) in U is said to be univalent in U if it assumes no value more than once in U. Denote by S, the subclass of A, of all univalent functions in U.

For $f(z)$ and $g(z)$ analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written, $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ with $w(z)=0$ and $|w(z)|<1$ in U such that $f(z) = g(w(z))$. That is if the range of one holomorphic function is contained in that of the second and these functions agree at a single point, then a sharp comparison of these two functions can be made.

The problem of finding sharp bounds for the linear functional $|a_3 - \zeta a_2^2|$ of any compact family of functions is popularly known as the Fekete-Szegő problem. This coefficient functional on the normalized analytic functions in the unit disk represents various geometric quantities. For example, for $\zeta = 1$, the functional represents Schwarzian derivative, which plays a significant role in the theory of univalent functions, conformal mapping and hypergeometric functions.

A function $f(z) \in A$ is said to be bi-univalent in U, if $f(z) \in S$ and its inverse has an analytic continuation to $|w| < 1$. The class of all bi-univalent functions is denoted by Σ.
The concept of bi-univalent functions was introduced by Lewin [18] who proved that if $f(z)$ is bi-univalent, then $|a_2| < 1.51$. Brannan and Clunie [10] improved Lewin's result to $|a_2| \leq \sqrt{2}$. There is a rich literature on the estimates of the initial coefficients of bi-univalent functions (see [11, 13, 16, 24, 25, 26]). However not much is known about the estimates of higher coefficients. It is well known that every function $f \in S$ has an inverse f^{-1} , satisfying $f^{-1}(f(z)) = z, (z \in U)$ and $f(f^{-1}(w)) = w, \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right)$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \quad (2)$$

Let K be simply connected, compact set in the Complex plane. Let *h* be analytic on K. It is possible to approximate *h* by polynomials uniformly on K called Faber polynomials, introduced by Faber [12]. These polynomials play an important role in geometric function theory.
A detailed discussion about Faber polynomial expansion for functions $f \in S$ of the form has been carried out in [[1], [2], [3]].

Geometric function theory provides a platform to have a multiple dimensional view on the different subclasses of analytic functions with help of q- calculus which is an effective tool of investigation. For example, the theory of q- calculus is used to describe the extension of the theory of univalent functions. For basic definitions, applications, terminologies, geometric properties and approximation one can refer [[5], [8], [9], [14], [17], [19], [20], [21]]. Let us suppose $0 < q < 1$ throughout this paper.

Definition. 1
The *q*-derivative of a function f is defined on a subset of $\mathbb{C}$ is given by

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \; if \;\; z \neq 0,$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.
Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

if f is differentiable. From (1), we have

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Where the symbol $[n]_q$ denotes the number

$$[n]_q = \frac{1-q^n}{1-q}$$

Definition. 2
The symmetric *q*-derivative $\tilde{D}_q f$ of a function f given by (1) is defined as follows:

$$(\tilde{D}_q f)(z) = \frac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z}, \quad if\ z \neq 0, \quad (3)$$

and $(\tilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.

From (3), we have the deduction

$$(\tilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad (4)$$

Where the symbol $[n]_q$ denotes the number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

From (2) and (4), we also have

$$(\tilde{D}_q g)(w) = \frac{g(qw) - g(q^{-1}w)}{(q-q^{-1})w} \quad (5)$$
$$= 1 - [2]_q a_2 w + [3]_q (2a_2^2 - a_3) w^2 - [4]_q (5a_2^3 - 5a_2 a_3 + a_4) w^3 + \cdots.$$

Lemma. 1 [6, 22]

If the function $p \in P$ is defined by

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots.$$

then

$$|p_n| \leq 2 (n \in \mathbb{N} = \{1, 2, 3, \cdots\}),$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

Let φ be an analytic function with positive real part in U, with $\varphi(0) = 1$ and $\varphi'(0) > 0$. Also, let $\varphi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Then, φ has the Taylor series expansion

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0). \quad (6)$$

2. Main Results

Definition. 3

Let $f \in A$. Then $f \in R_\Sigma(b, q, \varphi), b \in \mathbb{C} - \{0\}$ if $f \in \Sigma$,

$$Re\left(1 + \frac{1}{b}\left((D_q f)(z) - 1\right)\right) \prec \varphi(u) \quad (7)$$

and

$$Re\left(1 + \frac{1}{b}\left((D_q g)(w) - 1\right)\right) \prec \varphi(v) \quad (8)$$

Where $g = f^{-1}$.

2.1. COEFFICIENT BOUNDS FOR FUNCTIONS BELONGING TO THE CLASS $R_\Sigma(b, q, \varphi)$

Theorem 1

Let $f \in R_\Sigma(b, q, \varphi)$ and $g = f^{-1} \in R_\Sigma(b, q, \varphi)$. If $a_k = 0$ for $2 \leq k \leq n-1$ then

$$|a_n| \leq \frac{2|b|}{[n]_q}; n \geq 3.$$

Proof

Let $f \in R_{\Sigma}(b,q,\varphi)$ and $\varphi \in P$, then there exists two Schwarz functions $u(z) = c_1 z + c_2 z^2 + \cdots$ and $v(w) = d_1 w + d_2 w^2 + \cdots$ such that

$$1+\frac{1}{b}\left((D_q f)(z)-1\right) = \varphi(u(z)) \quad (9)$$

$$1+\frac{1}{b}\left((D_q g)(w)-1\right) = \varphi(v(w)) \quad (10)$$

Where

$$\varphi(u(z)) = 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\varphi_k D_n^k(c_1, c_2, \cdots c_n)z^n \quad (11)$$

and

$$\varphi(v(w)) = 1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\varphi_k D_n^k(d_1, d_2, \cdots d_n)w^n \quad (12)$$

From (9) and (11) we have

$$\frac{[n]_q a_n}{b} = \sum_{k=1}^{n-1}\varphi_k D_n^k(c_1, c_2, \cdots c_n), n \geq 2. \quad (13)$$

From (10) and (12), we have

$$\frac{[n]_q b_n}{b} = \sum_{k=1}^{n-1}\varphi_k D_n^k(d_1, d_2, \cdots d_n), n \geq 2. \quad (14)$$

For $a_k = 0$ for $2 \leq k \leq n-1$, (13) and (14) respectively yield

$$\frac{[n]_q a_n}{b} = \varphi_1 c_{n-1}$$

and

$$\frac{[n]_q b_n}{b} = -\frac{[n]_q a_n}{b} = \varphi_1 d_{n-1}$$

By definition of K_n^p we have $b_n = -a_n$.

Upon simplification, we obtain

$$a_n = \frac{b}{[n]_q}\varphi_1 c_{n-1} \quad (15)$$

$$a_n = -\frac{b}{[n]_q}\varphi_1 d_{n-1} \quad (16)$$

Taking the absolute values of (15) and (16) and using the facts that $|\varphi_1| \leq 2$, $|c_{n-1}| \leq 1$ and $|d_{n-1}| \leq 1$, we obtain

$$|a_n| \leq \frac{2|b|}{[n]_q}$$

Remark 1

If $q \to 1^-$ then the above theorem reduces to the results of Hamidi and Jahangiri [15].

Remark 2

If $|b| = 1$ then above theorem reduces to the results of Altinkaya and Yalcin [7] (Theorem 7 for p=1).

Theorem 2

Let $f \in R_\Sigma(b,q,\varphi)$ and $g = f^{-1} \in R_\Sigma(b,q,\varphi)$. Then

$$\text{(i)} \ |a_2| \le \begin{cases} \dfrac{2q|b|}{q^2+1}, & |b| < \dfrac{q^4+2q^2+1}{q^4+q^2+1} \\[2ex] \dfrac{2q\sqrt{|b|}}{\sqrt{q^4+q^2+1}}, & |b| \ge \dfrac{q^4+2q^2+1}{q^4+q^2+1}. \end{cases}$$

$$\text{(ii)} \ |a_3| \le \begin{cases} \dfrac{4q^2|b|^2}{(q^2+1)^2} + \dfrac{2q^2|b|}{q^4+q^2+1}, & |b| < \dfrac{(q^2+1)^2}{2(q^4+q^2+1)} \\[2ex] \dfrac{4q^2|b|}{q^4+q^2+1}, & |b| \ge \dfrac{(q^2+1)^2}{2(q^4+q^2+1)}. \end{cases}$$

$$\text{(iii)} \ |a_3 - \mu a_2^2| \le \frac{2\mu q^2|b|}{q^4+q^2+1}, \mu = 1, \frac{3}{2}, 2.$$

Proof

Letting $n=2$ and $n=3$ in [13] and [14] respectively, we get

$$\frac{[2]_q a_2}{b} = \varphi_1 c_1 \quad (17)$$

$$\frac{[3]_q a_3}{b} = \varphi_1 c_2 + \varphi_2 c_1^2 \quad (18)$$

and

$$\frac{[2]_q b_2}{b} = \varphi_1 d_1 \quad (19)$$

$$\frac{[3]_q b_3}{b} = \varphi_1 d_2 + \varphi_2 d_1^2 \quad (20)$$

Comparing (5) with (19) and (20)

$$\frac{-[2]_q a_2}{b} = \varphi_1 d_1 \quad (21)$$

$$\frac{[3]_q (2a_2^2 - a_3)}{b} = \varphi_1 d_2 + \varphi_2 d_1^2 \quad (22)$$

Using $c_1 = -d_1$ in either of (17) and (21), we deduce

$$|a_2| \le \frac{2|b|}{[\tilde{2}]_q} = \frac{2q|b|}{q^2+1}$$

From (18) and (22), we get

$$\frac{2[3]_q a_2^2}{b} = \varphi_1(c_2+d_2) + \varphi_2(c_1^2+d_1^2)$$

and thus

$$|a_2| \le \sqrt{\frac{4|b|}{[3]_q}} = \frac{2q\sqrt{|b|}}{\sqrt{q^4+q^2+1}}$$

Now the bounds for $|a_2|$ are justified since $|a_2| \le \sqrt{\frac{4|b|}{[3]_q}} = \frac{2q\sqrt{|b|}}{\sqrt{q^4+q^2+1}}$ for $|b| < \frac{q^4+2q^2+1}{q^4+q^2+1}$.

From (18), we get

$$|a_3| = \frac{|b|(|\varphi_1 c_2 + \varphi_2 c_1^2|)}{[3]_q} \le \frac{4|b|}{[3]_q} = \frac{4q^2|b|}{q^4+q^2+1} \quad (23)$$

On the other hand subtracting (22) from (18).

$$\frac{2[3]_q}{b}(a_3 - a_2^2) = \varphi_1(c_2 - d_2). \quad (24)$$

Solving the above equation for a_3 and taking absolute value

$$|a_3| \le \frac{4q^2|b|^2}{(q^2+1)^2} + \frac{2q^2|b|}{q^4+q^2+1} \quad (25)$$

Now, Theorem 2.2 (ii) follows from (23) and (25) upon noticing that

$$\frac{4q^2|b|^2}{(q^2+1)^2} + \frac{2q^2|b|}{q^4+q^2+1} < \frac{4|b|q^2}{q^4+q^2+1} \text{ if } |b| < \frac{(q^2+1)^2}{2(q^4+q^2+1)}$$

For the third part of the theorem, we rewrite (24) as

$$a_3 - a_2^2 = \frac{b}{2[3]_q}(\varphi_1(c_2 - d_2)) \quad (26)$$

Taking absolute values, we get

$$|a_3 - a_2^2| = \frac{|b||\varphi_1(c_2 - d_2)|}{2[3]_q} \le \frac{2|b|q^2}{q^4+q^2+1}$$

We rewrite (22) as

$$2a_2^2 - a_3 = \frac{b}{[3]_q}(\varphi_1 d_2 + \varphi_2 d_1^2) \quad (27)$$

Taking absolute values, we get

$$|a_3 - 2a_2^2| = \frac{|b|q^2}{q^4+q^2+1}|\varphi_1 d_2 + \varphi_2 d_1^2| \le \frac{4|b|q^2}{q^4+q^2+1}$$

Adding (26) and (27) and taking absolute value,

$$|a_2 - \frac{3}{2}a_2^2| \le \frac{3|b|q^2}{q^4+q^2+1}.$$

2.2. FEKETE- SZEGÖ INEQUALITY FOR FUNCTIONS BELONGING TO THE CLASS $\mathrm{R}_\Sigma(b,q,\varphi)$

Theorem 3

Let f given by (1) be in the class $\mathrm{R}_\Sigma(b,q,\varphi)$ and $\zeta \in \mathbb{R}$. Then

$$|a_3-\zeta a_2^2|\le\begin{cases}\dfrac{B_1|b|}{4[3]_q} & for\ 0\le|h(\zeta)|\le\dfrac{1}{4[3]_q}\\[2ex] 4B_1|h(\zeta)| & for\ |h(\zeta)|\ge\dfrac{1}{4[3]_q}.\end{cases}$$

where $h(\zeta)=\dfrac{B_1^2(1-\zeta)}{4[b[3]_qB_1^2+[2]_q^2(B_1-B_2)]}$

Proof

Let $f\in \mathrm{R}_\Sigma(b,q,\varphi)$ and g be the analytic extension of f^{-1} to U then there exists two functions u and v, analytic in U with $u(0)=v(0)=0$, $|u(z)|<1, |v(w)|<1$ and $z,w\in U$ such that

$$1+\frac{1}{b}\left((\tilde{D}_q f)(z)-1\right)=\varphi(u(z))\ (28)$$

$$1+\frac{1}{b}\left((\tilde{D}_q g)(w)-1\right)=\varphi(v(w))\ (29)$$

where $g=f^{-1}$.

Next, define the functions $p,q\in P$ by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots\ (30)$$

$$q(z)=\frac{1+v(w)}{1-v(w)}=1+q_1w+q_2w^2+\cdots\ (31)$$

From the above definitions, one can derive

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2z^2\right)+\cdots\ (32)$$

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}q_1w+\frac{1}{2}\left(q_2-\frac{1}{2}q_1^2w^2\right)+\cdots\ (33)$$

Combining (6), (28), (29), (32) and (33)

$$1+\frac{1}{b}\left((\tilde{D}_q f)(z)-1\right)=1+\frac{1}{2}B_1p_1z+\left(\frac{1}{4}B_2p_1^2+\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)\right)z^2+\cdots\ (34)$$

$$1+\frac{1}{b}\left((\tilde{D}_q g)(w)-1\right)=1+\frac{1}{2}B_1q_1w+\left(\frac{1}{4}B_2q_1^2+\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)\right)w^2+\cdots\ (35)$$

From (34) and (35), we deduce

$$\frac{[2]_qa_2}{b}=\frac{1}{2}B_1p_1\ (36)$$

$$\frac{[3]_qa_3}{b}=\frac{1}{4}B_2p_1^2+\frac{1}{2}B_1\left(p_2-\frac{1}{2}p_1^2\right)(37)$$

and

$$-\frac{[2]_qa_2}{b}=\frac{1}{2}B_1q_1\ (38)$$

$$\frac{[3]_q(2a_2^2-a_3)}{b}=\frac{1}{4}B_2q_1^2+\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right) \quad (39)$$

From (36) and (38), we get

$$p_1=-q_1 \quad (40)$$

Subtracting (37) from (39) and applying (40)

$$a_3=a_2^2+\frac{bB_1}{4[3]_q}(p_2-q_2) \quad (41)$$

By adding (37) to (39), we get

$$\frac{2[3]_q}{b}a_2^2=\frac{1}{2}B_1(p_2+q_2)+\frac{1}{2}p_1^2(B_2-B_1).$$

Using (36) and (38)

$$a_2^2=\frac{bB_1^3(p_2+q_2)}{4\left[b[3]_qB_1^2+[2]_q^2(B_1-B_2)\right]} \quad (42)$$

From (41) and (42), we get

$$a_3-\zeta a_2^2=bB_1\left[\left(h(\zeta)+\frac{1}{4[3]_q}\right)p_2+\left(h(\zeta)-\frac{1}{4[3]_q}\right)q_2\right]$$

Where

$$h(\zeta)=\frac{B_1^2(1-\zeta)}{4[b[3]_qB_1^2+[2]_q^2(B_1-B_2)]}$$

Then, by Lemma 1 and (6)

$$|a_3-\zeta a_2^2|\le\begin{cases}\dfrac{|B_1||b|}{4[3]_q} & for\ 0\le|h(\zeta)|\le\dfrac{1}{4[3]_q}\\[2ex] 4|B_1||h(\zeta)| & for\ |h(\zeta)|\ge\dfrac{1}{4[3]_q}.\end{cases}$$

Corollary 1 If $f\in \mathrm{R}_\Sigma(b,q,\varphi)$ then taking $\zeta=1$, we get

$$|a_3-a_2^2|\le\frac{q^2|b|B_1}{4[q^4+q^2+1]}. \quad (43)$$

Corollary 2 Let

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^\beta=1+2\beta z+2\beta^2z^2+\cdots,(0<\beta\le 1).$$

then from (43), we have

$$|a_3-a_2^2|\le\frac{q^2|b|\beta}{2[q^4+q^2+1]}.$$

Corollary 3 Let

$$\varphi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)z^2+\cdots,(0\le\beta<1).$$

then the inequality (43) reduces to

$$| a_3 - a_2^2 | \leq \frac{q^2 | b | (1-\beta)}{2[q^4 + q^2 + 1]}.$$

3. Conclusion

We have estimated the bounds for the coefficients and also the linear functional which is popularly known as Fekete- Szegö problem, for functions belonging to the class defined in this article. We also have seen our results reducing to the results discussed in various other articles.

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