Coefficient Estimates and Fekete-Szegő Inequality for a Subclass of Bi-Univalent Functions Defined by Symmetric Q-Derivative Operator by Using Faber Polynomial Techniques

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1. Introduction

Let A be the class of all normalized functions of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \] (1)

which are analytic in the Unit disk U.

A function that is regular (holomorphic) in U is said to be univalent in U if it assumes no value more than once in U. Denote by S, the subclass of A, of all univalent functions in U.

For \( f(z) \) and \( g(z) \) analytic in U, we say that \( f(z) \) is subordinate to \( g(z) \), written \( f(z) \prec g(z) \), if there exists a Schwarz function \( w(z) \) with \( w(z) = 0 \) and \( |w(z)| < 1 \) in U such that \( f(z) = g(w(z)) \). That is if the range of one holomorphic function is contained in that of the second and these functions agree at a single point, then a sharp comparison of these two functions can be made.

The problem of finding sharp bounds for the linear functional \( |a_n - \zeta a_2^2| \) of any compact family of functions is popularly known as the Fekete-Szegő problem. This coefficient functional on the normalized analytic
functions in the unit disk represents various geometric quantities. For example, for \( \zeta = 1 \), the functional represents Schwarzian derivative, which plays a significant role in the theory of univalent functions, conformal mapping and hypergeometric functions.

A function \( f(z) \in A \) is said to be bi-univalent in \( U \), if \( f(z) \in S \) and its inverse has an analytic continuation to \( |w| < 1 \). The class of all bi-univalent functions is denoted by \( \Sigma \).

The concept of bi-univalent functions was introduced by Lewin [18] who proved that if \( f(z) \) is bi-univalent, then \( \alpha_2 \leq 1.51 \). Brannan and Clunie [10] improved Lewin's result to \( |\alpha_2| \leq \sqrt{2} \). There is a rich literature on the estimates of the initial coefficients of bi-univalent functions (see [11, 13, 16, 24, 25, 26]). However not much is known about the estimates of higher coefficients. It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), satisfying \( f^{-1}(f(z)) = z, (z \in U) \) and \( f(f^{-1}(w)) = w \), \( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \), where

\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_1)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{2}
\]

Let K be simply connected, compact set in the Complex plane. Let \( h \) be analytic on K. It is possible to approximate \( h \) by polynomials uniformly on K called Faber polynomials, introduced by Faber [12]. These polynomials play an important role in geometric function theory.

A detailed discussion about Faber polynomial expansion for functions \( f \in S \) of the form has been carried out in [1], [2], [3].

Geometric function theory provides a platform to have a multiple dimensional view on the different subclasses of analytic functions with help of \( q \)-calculus which is an effective tool of investigation. For example, the theory of \( q \)-calculus is used to describe the extension of the theory of univalent functions. For basic definitions, applications, terminologies, geometric properties and approximation one can refer [5], [8], [9], [14], [17], [19], [20], [21]]. Let us suppose \( 0 < q < 1 \) throughout this paper.

**Definition. 1**

The \( q \)-derivative of a function \( f \) is defined on a subset of \( \mathbb{D} \) is given by

\[
(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \text{ if } z \neq 0,
\]

and \( (D_q f)(0) = f'(0) \) provided \( f'(0) \) exists.

Note that

\[
\lim_{q \to 1} (D_q f)(z) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)
\]

if \( f \) is differentiable. From (1), we have

\[
(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.
\]

Where the symbol \([n]_q \) denotes the number

\[
[n]_q = \frac{1-q^n}{1-q}
\]

**Definition. 2**

The symmetric \( q \)-derivative \( \tilde{D}_q f \) of a function \( f \) given by (1) is defined as follows:
\( (\tilde{D}_q f)(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z} , \quad \text{if } z \neq 0, \quad (3) \)

and \( (\tilde{D}_q f)(0) = f'(0) \) provided \( f'(0) \) exists.

From (3), we have the deduction

\[
(\tilde{D}_q f)(z) = 1 + \sum_{n=2}^\infty [n]_q a_n z^{n-1},
\]

Where the symbol \([n]_q\) denotes the number

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

From (2) and (4), we also have

\[
(\tilde{D}_q g)(w) = \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})w} = 1 - [2]_q a_2 w + [3]_q (2a_2^2 - a_0)w^2 - [4]_q (5a_2^3 - 5a_2a_3 + a_4)w^3 + \cdots.
\]

Lemma 1 [6, 22]

If the function \( p \in P \) is defined by

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots.
\]

then

\[
|p_n| \leq 2(n \in \{1, 2, 3, \cdots\}),
\]

and

\[
|p_2 - \frac{p_1^2}{2}| \leq 2 - \frac{|p_1|^2}{2}.
\]

Let \( \varphi \) be an analytic function with positive real part in \( U \), with \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \). Also, let \( \varphi(U) \) be starlike with respect to 1 and symmetric with respect to the real axis. Then, \( \varphi \) has the Taylor series expansion

\[
\varphi(z) = 1 + B_2 z + B_2 z^2 + B_3 z^3 + \cdots (B_i > 0). \quad (6)
\]

2. Main Results

Definition 3

Let \( f \in A \). Then \( f \in R_{\varphi}(b, q, \varphi), b \in \{1 \} - \{0\} \) if \( f \in \Sigma \),

\[
\text{Re} \left( 1 + \frac{1}{b} ((D_q f)(z) - 1) \right) < \varphi(u) \quad (7)
\]

and

\[
\text{Re} \left( 1 + \frac{1}{b} ((D_q g)(w) - 1) \right) < \varphi(v) \quad (8)
\]

Where \( g = f^{-1} \).

2.1. COEFFICIENT BOUNDS FOR FUNCTIONS BELONGING TO THE CLASS \( R_{\varphi}(b, q, \varphi) \)

Theorem 1

Let \( f \in R_{\varphi}(b, q, \varphi) \) and \( g = f^{-1} \in R_{\varphi}(b, q, \varphi) \). If \( a_k = 0 \) for \( 2 \leq k \leq n - 1 \) then

\[
a_1 \leq \frac{2}{[n]_q}; n \geq 3.
\]
Proof
Let \( f \in \mathbb{R}_2(b,q,\varphi) \) and \( \varphi \in \mathbb{P} \), then there exists two Schwarz functions \( u(z) = c_1z + c_2z^2 + \cdots \) and \( v(w) = d_1w + d_2w^2 + \cdots \) such that

\[
1 + \frac{1}{b} \left( (D_q f)(z) - 1 \right) = \varphi(u(z)) \quad (9)
\]

\[
1 + \frac{1}{b} \left( (D_q g)(w) - 1 \right) = \varphi(v(w)) \quad (10)
\]

Where

\[
\varphi(u(z)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \varphi_k D_n^k (c_1, c_2, \cdots c_n) z^n \quad (11)
\]

and

\[
\varphi(v(w)) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \varphi_k D_n^k (d_1, d_2, \cdots d_n) w^n \quad (12)
\]

From (9) and (11) we have

\[
\frac{[n]_q a_n}{b} = \sum_{k=1}^{n-1} \varphi_k D_n^k (c_1, c_2, \cdots c_n), n \geq 2. \quad (13)
\]

From (10) and (12), we have

\[
\frac{[n]_q b_n}{b} = \sum_{k=1}^{n-1} \varphi_k D_n^k (d_1, d_2, \cdots d_n), n \geq 2. \quad (14)
\]

For \( a_k = 0 \) for \( 2 \leq k \leq n-1 \), (13) and (14) respectively yield

\[
\frac{[n]_q a_n}{b} = \varphi_1 c_{n-1}
\]

and

\[
\frac{[n]_q b_n}{b} = -\frac{[n]_q a_n}{b} = \varphi_1 d_{n-1}
\]

By definition of \( K_p^\rho \) we have \( b_n = -a_n \).

Upon simplification, we obtain

\[
a_n = \frac{b}{[n]_q} \varphi_1 c_{n-1} \quad (15)
\]

\[
a_n = -\frac{b}{[n]_q} \varphi_1 d_{n-1} \quad (16)
\]

Taking the absolute values of (15) and (16) and using the facts that \( |\varphi_1| \leq 2 \), \( |c_{n-1}| \leq 1 \), and \( |d_{n-1}| \leq 1 \), we obtain

\[
|a_n| \leq \frac{2 |b|}{[n]_q}
\]

Remark 1
If \( q \to 1^- \) then the above theorem reduces to the results of Hamidi and Jahangiri [15].

Remark 2
If \( \mathfrak{h}_q = 1 \) then above theorem reduces to the results of Altinkaya and Yalcin [7] (Theorem 7 for \( p=1 \)).
Theorem 2

Let \( f \in \mathbb{R}_\mathbb{Z}(b,q,\varphi) \) and \( g = f^{-1} \in \mathbb{R}_\mathbb{Z}(b,q,\varphi) \). Then

\[
(i) \quad a_j \leq \begin{cases} \frac{2\sqrt{q} h}{q^2 + 1}, & |h| < \frac{q^4 + 2q^2 + 1}{q^4 + q^2 + 1} \\ \frac{2\sqrt{q} h}{\sqrt{q^4 + q^2 + 1}}, & |h| \geq \frac{q^4 + 2q^2 + 1}{q^4 + q^2 + 1} \end{cases}
\]

\[
(ii) \quad a_j \leq \begin{cases} \frac{4q^\frac{3}{2} h}{(q^2 + 1)^2} + \frac{2q^\frac{1}{2} h}{q^4 + q^2 + 1}, & |h| < \frac{(q^2 + 1)^2}{2(q^4 + q^2 + 1)} \\ \frac{4q^\frac{3}{2} h}{q^4 + q^2 + 1}, & |h| \geq \frac{(q^2 + 1)^2}{2(q^4 + q^2 + 1)} \end{cases}
\]

\[
(iii) \quad a_j - \mu a_j \leq \frac{2\mu q^\frac{3}{2} h}{q^4 + q^2 + 1}, \mu = 1, \frac{3}{2}, 2.
\]

Proof

Letting \( n = 2 \) and \( n = 3 \) in [13] and [14] respectively, we get

\[
\frac{[2]q^1 a_2}{b} = \varphi_1 c_1, \quad (17)
\]

\[
\frac{[3]q^2 a_3}{b} = \varphi_1 c_1 + \varphi_2 c_1^2, \quad (18)
\]

and

\[
\frac{[2]q^1 b_2}{b} = \varphi_1 d_1, \quad (19)
\]

\[
\frac{[3]q^2 b_3}{b} = \varphi_1 d_2 + \varphi_2 d_1^2, \quad (20)
\]

Comparing (5) with (19) and (20)

\[
\frac{-[2]q^1 a_2}{b} = \varphi_1 d_1, \quad (21)
\]

\[
\frac{[3]q^2 (2a_2 - a_3)}{b} = \varphi_1 d_2 + \varphi_2 d_1^2, \quad (22)
\]

Using \( c_1 = -d_1 \) in either of (17) and (21), we deduce

\[
|a_2| \leq 2 \left| \frac{b}{[2]q^1} \right| = \frac{2q |b|}{q^2 + 1}
\]

From (18) and (22), we get

\[
\frac{2[3]q^2 a_2^2}{b} = \varphi_1 (c_2 + d_2) + \varphi_2 (c_1^2 + d_1^2)
\]

and thus
\[ |a_2| \leq \frac{4|b|}{\sqrt{3q}} = \frac{2q\sqrt{|b|}}{\sqrt{q^4 + q^2 + 1}} \]

Now the bounds for \(|a_2|\) are justified since
\[ |a_2| \leq \frac{4|b|}{\sqrt{3q}} = \frac{2q\sqrt{|b|}}{\sqrt{q^4 + q^2 + 1}} \quad \text{for} \quad |b| < \frac{q^4 + 2q^2 + 1}{q^4 + q^2 + 1}. \]

From (18), we get
\[ |a_3| = \frac{|b|(|\varphi_1^2 + \varphi_2^2|)}{[3]_q} \leq \frac{4|b|}{[3]_q} = \frac{4q^2|b|}{q^4 + q^2 + 1} \quad (23) \]

On the other hand subtracting (22) from (18).
\[ \frac{2[3]_q}{b} (a_3 - a_2^2) = \varphi_1(c_2 - d_2). \quad (24) \]

Solving the above equation for \(a_3\) and taking absolute value
\[ |a_3| \leq \frac{4q^2|b|^2}{(q^2 + 1)^2} + \frac{2q^2|b|}{q^4 + q^2 + 1} \quad (25) \]

Now, Theorem 2.2 (ii) follows from (23) and (25) upon noticing that
\[ \frac{4q^2|b|^2}{(q^2 + 1)^2} + \frac{2q^2|b|}{q^4 + q^2 + 1} < \frac{4|b|q^2}{q^4 + q^2 + 1} \text{ if } |b| < \frac{(q^2 + 1)^2}{2(q^4 + q^2 + 1)} \]

For the third part of the theorem, we rewrite (24) as
\[ a_3 - a_2^2 = \frac{b}{2[3]_q} (\varphi_1(c_2 - d_2)) \quad (26) \]

Taking absolute values, we get
\[ |a_3 - a_2^2| = \frac{|b|\varphi_1(c_2 - d_2)}{2[3]_q} \leq \frac{2|b|q^2}{q^4 + q^2 + 1} \]

We rewrite (22) as
\[ 2a_2^2 - a_3 = \frac{b}{[3]_q} (\varphi_1d_2 + \varphi_2d_1^2) \quad (27) \]

Taking absolute values, we get
\[ |a_3 - 2a_2^2| = \frac{|b|q^2}{q^4 + q^2 + 1} |\varphi_1d_2 + \varphi_2d_1^2| \leq \frac{4|b|q^2}{q^4 + q^2 + 1} \]

Adding (26) and (27) and taking absolute value,
\[ |a_2 - \frac{3}{2}a_2^2| \leq \frac{3|b|q^2}{q^4 + q^2 + 1}. \]

2.2. FEKETE- SZEGÖ INEQUALITY FOR FUNCTIONS BELONGING TO THE CLASS \( R_\zeta(b, q, \varphi) \)

Theorem 3
Let \( f \) given by (1) be in the class \( R_\zeta(b, q, \varphi) \) and \( \zeta \in \mathbb{C} \). Then
\[
\begin{align*}
\alpha_j - \zeta \alpha_j^2 & \leq \begin{cases} 
\frac{B_j}{4[3]_q} & \text{for } 0 < h(\zeta) \leq \frac{1}{4[3]_q} \\
4B_j & \text{for } h(\zeta) \geq \frac{1}{4[3]_q}.
\end{cases}
\end{align*}
\]

where \( h(\zeta) = \frac{B_j^2(1 - \zeta)}{4[b[3]_qB_1^2 + [2\zeta_q(B_1 - B_2)]} \)

**Proof**

Let \( f \in \mathbb{R}_+ (b, q, \varphi) \) and \( g \) be the analytic extension of \( f^{-1} \) to \( U \) then there exists two functions \( u \) and \( v \), analytic in \( U \) with \( u(0) = v(0) = 0 \), \( |u(z)| < 1 \), \( |v(w)| < 1 \) and \( z, w \in U \) such that

\[
1 + \frac{1}{b}((\tilde{D}_q f)(z) - 1) = \varphi(u(z)) \quad (28)
\]

\[
1 + \frac{1}{b}((\tilde{D}_q g)(w) - 1) = \varphi(v(w)) \quad (29)
\]

where \( g = f^{-1} \).

Next, define the functions \( p, q \in P \) by

\[
p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + p_z z + p_z z^2 + \cdots \quad (30)
\]

\[
q(z) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_z w + q_z w^2 + \cdots \quad (31)
\]

From the above definitions, one can derive

\[
u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} p_z z + \frac{1}{2} \left( p_z - \frac{1}{2} p_z^2 \right) z^2 + \cdots \quad (32)
\]

\[
v(w) = \frac{q(w) - 1}{q(w) + 1} = \frac{1}{2} q_z w + \frac{1}{2} \left( q_z - \frac{1}{2} q_z^2 \right) w^2 + \cdots \quad (33)
\]

Combining (6), (28), (29), (32) and (33)

\[
1 + \frac{1}{b}((\tilde{D}_q f)(z) - 1) = 1 + \frac{1}{2} B_z p_z z + \frac{1}{4} B_z p_z^2 + \frac{1}{2} B_z \left( p_z - \frac{1}{2} p_z^2 \right) z^2 + \cdots \quad (34)
\]

\[
1 + \frac{1}{b}((\tilde{D}_q g)(w) - 1) = 1 + \frac{1}{2} B_q q_z w + \frac{1}{4} B_q q_z^2 + \frac{1}{2} B_q \left( q_z - \frac{1}{2} q_z^2 \right) w^2 + \cdots \quad (35)
\]

From (34) and (35), we deduce

\[
\frac{[2]_q a_z}{b} = \frac{1}{2} B_z p_z \quad (36)
\]

\[
\frac{[3]_q a_z}{b} = \frac{1}{4} B_z p_z^2 + \frac{1}{2} B_z \left( p_z - \frac{1}{2} p_z^2 \right) \quad (37)
\]

and

\[
-\frac{[2]_q a_z}{b} = \frac{1}{2} B_q q_z \quad (38)
\]
\[ \frac{[3]_q(2a_2^2 - a_1)}{b} = \frac{1}{4} B_i q_i^2 + \frac{1}{2} B_i \left( q_2 - \frac{1}{2} q_i^2 \right) \]  

(39)

From (36) and (38), we get

\[ p_i = -q_i \]  

(40)

Subtracting (37) from (39) and applying (40)

\[ a_3 = a_2^2 + \frac{b B_i}{4[3]_q} (p_2 - q_2) \]  

(41)

By adding (37) to (39), we get

\[ \frac{2[3]_q}{b} a_2^2 = \frac{1}{2} B_i (p_2 + q_2) + \frac{1}{2} p_i^2 (B_2 - B_1). \]

Using (36) and (38)

\[ a_2^2 = \frac{b B_i^3 (p_2 + q_2)}{4\left[ b [3]_q B_i^2 + [2]_q^2 (B_i - B_2) \right]} \]  

(42)

From (41) and (42), we get

\[ a_3 - \zeta a_2^2 = b B_i \left[ h(\zeta) + \frac{1}{4[3]_q} p_2 + \left( h(\zeta) - \frac{1}{4[3]_q} \right) q_2 \right] \]

Where

\[ h(\zeta) = \frac{B_i^2 (1 - \zeta)}{4[b [3]_q B_i^2 + [2]_q^2 (B_i - B_2)]} \]

Then, by Lemma 1 and (6)

\[ a_3 - \zeta a_2^2 \leq \begin{cases} \frac{b}{4[3]_q} & \text{for } 0 \leq h(\zeta) \leq \frac{1}{4[3]_q} \\ \frac{4 B_i}{h(\zeta)} & \text{for } h(\zeta) \geq \frac{1}{4[3]_q}. \end{cases} \]

**Corollary 1** If \( f \in \mathbb{R}_2(b, q, \phi) \) then taking \( \zeta = 1 \), we get

\[ | a_3 - a_2^2 | \leq q^2 \frac{|b| B_i}{4q^4 + q^2 + 1}, \]  

(43)

**Corollary 2** Let

\[ \phi(z) = \left( \frac{1 + z}{1 - z} \right)^\beta = 1 + 2\beta z + 2\beta^2 z^2 + \ldots, \quad (0 < \beta \leq 1). \]

then from (43), we have

\[ | a_3 - a_2^2 | \leq q^2 \frac{|b| \beta}{2[q^4 + q^2 + 1]}. \]

**Corollary 3** Let

\[ \phi(z) = \frac{1 + (1 - 2\beta) z}{1 - z} = 1 + 2(1 - \beta) z + 2(1 - \beta) z^2 + \ldots, \quad (0 \leq \beta < 1). \]

then the inequality (43) reduces to
\[ |a_3 - a_2^2| \leq \frac{q^2 |b| (1-\beta)}{2[q^4 + q^2 + 1]} \]

3. Conclusion

We have estimated the bounds for the coefficients and also the linear functional which is popularly known as Fekete-Szego problem, for functions belonging to the class defined in this article. We also have seen our results reducing to the results discussed in various other articles.

References


