ISSN 2303-4521
Periodicals of Engineering and Natural Sciences *Original Research*
Vol. 10, No. 1, January 2022, pp.330-342

On a new double weighted exponential-pareto distribution: Properties and estimation

Rawaa Salh AL-Saffar [1], Mahdi Wahhab Neamah [2]
[1]Department of Statistics, Faculty of Administration and Economics, Al-Mustansiriyah University, Baghdad -Iraq
[2] Departments of Statistics, Faculty of Administration and Economics, University of Karbala, Kerbala, Iraq

ABSTRACT

The main aim of this manuscript to present a new weighted distribution named the Double Weighted Exponential Pareto Distribution (DWEOD). This paper constructed and studied this new distribution. The quantifiable properties are discussed, including the mean, variance, harmonic mean, coefficient of variation, reliability function, moments generating function, mode, hazard function, and the reverse hazard function. Moreover, this work estimated the parameters of this distribution by the maximum likelihood estimation method and the method of the moment.

Keywords: Double Weighted distribution, Exponential Pareto distribution, MLE, MOM, Percentiles.

Corresponding Author:

Rawaa Salh AL-Saffar
Department of Statistics, Faculty of Administration and Economics
Al-Mustansiriyah University
Baghdad -Iraq
rawaaalsaffar@uomustansiriyah.edu.iq

1. Introduction

Recently, researchers discussed various topics in statistics [1, 2] or more precisely in new distribution [3]. The Weighted Distribution Principle offers an approach to model specification and data interpretation issues. Fisher [4] examined how methods of ascertainment can impact the type of distribution of recorded perception, and it was introduced and formulated in general terms in (1965) by Rao in connection with modeling statistical data where it was not found to be appropriate to the normal practice of using the standard distribution for the purpose [5]. He defined numerous circumstances that relate to instances where it is not possible to treat the reported outcomes as a random sample from the original distribution, but it could be by weighted distribution modeling. The literature of the exponential Pareto distribution absorbs in most of the analysts due to its broad range application [6]. The definition and the concepts of double weighted distribution proposed first time by Al-Khadim and Hantoosh [7, 8], applies it to the exponential distribution and derives the statistical properties for the Double Weighted Exponential distribution. The Double weighted Rayleigh distribution properties were discussed by Nasr [9] and estimation is developed and considered. The statistical features and properties are discussed and acquired, such as the mean, harmonic mean, mode, variance, coefficient of variation, moment, coefficient of skewness, coefficient of kurtosis, hazard function, reverse hazard function and reliability function. Two different estimations methods used to estimate this distribution: the maximum likelihood estimation method, and the method of the moment. In (2014), Ahmed, apply it to the characterization and estimation of Double weighted Rayleigh distribution and its properties [10]. The statistical properties of the modified Double weighted exponential distribution are discovered by Khadim [11]. A better-fitted probability model has been chosen by using the Kolomogorov – Imirnov test or Beta-Invers [12, 13]. The Weibull-Rayleigh distribution utilized and demonstrated its application using lifetime data. More recently, Basheer used alpha power inverse Weibull distribution and receive the p.d.f, c.d.f, reliability, hazard, and revers hazard function [14]. Saghir and Saleen studied and conversed the statistical properties of the Double weighted Weibull distribution, including the mean, variance, Reliability function. The MLE estimation method is used to estimate this distribution. By applying it to real-life data, the utility of the distribution has been demonstrated [15].

Suppose X is a non-negative random variable with probability density function (pdf) f(x), then the pdf of the weighted random variable Xw is given by

$$f_w(x) = \frac{W(x)f(x)}{\int x\, f(x)dx} \quad , \qquad x > 0$$

Where W(x)= x be a non-negative weight function.

Depending upon the choice of the weight function $W(x)$, we have different weighted models. Clearly when $W(x) = x$, the resulting distribution is called length-biased whose pdf is given by:

$$f_i(x) = \frac{x\, f(x)}{E(x)} \quad , \qquad x > 0$$

This paper introduced the Double Weighted distribution (DWD), which takes one form or type of weight function $W(x) = x$, and using the Exponential Pareto distribution as original distribution, this work derives also the pdf and some useful properties of Double weighted Exponential Pareto distribution.

Double Weighted Exponential Pareto Distribution (DWEPD)

The Double Weighted distribution is given by:

$$f_w(x, C) = \frac{W(x)f(x)F(cx)}{\int_0^\infty W(x)f(x)F(cx)dx} \quad , \qquad x \geq 0\, , C > 0$$

$$WD=\int_0^\infty W(x)f(x)F(cx)\, dx$$

Where $w(x)$ is the first weight and the second is (cx). $F(cx)$ varies with on the original distribution f(x).

Now by considering the first weighted function w(x)=x and probability density function (pdf) of Exponential Pareto distribution are given by:

$$f\,(x,\theta,p,\lambda)=\frac{\theta\lambda}{p}\,(\frac{x}{p})^{\theta-1}\, e^{-\lambda(\frac{x}{p})^{\theta}} \; , x>0 \, , x>p \qquad \ldots(1)$$

$$F(cx,\theta,p,\lambda)=1 - e^{-\lambda(\frac{cx}{p})^{\theta}} \qquad , c > 0$$

and $W_D = \int_0^\infty W(x)f(x)F(cx)\, dx = \int_0^\infty x\, \frac{\theta\lambda}{p}\, (\frac{x}{p})^{\theta-1} e^{-\lambda(\frac{x}{p})^{\theta}}\, [\, 1 - e^{-\lambda(\frac{cx}{p})^{\theta}}\,]dx$

$$\frac{\theta\lambda}{p^\theta}[\int_0^\infty x^\theta e^{-\lambda(\frac{x}{p})^\theta} dx - \int_0^\infty x^\theta e^{-\lambda(\frac{x}{p})^\theta}\, e^{-\lambda(\frac{cx}{p})^\theta} dx$$

$$= \frac{\theta\lambda}{p^\theta}[\int_0^\infty x^\theta e^{-\lambda(\frac{x}{p})^\theta} dx - \int_0^\infty x^\theta e^{-\lambda(\frac{x}{p})^\theta\, [1+C^\theta)} dx$$

Let the first integration $Z= \lambda(\frac{x}{p})^\theta$, $x=\frac{p}{\lambda^{\frac{1}{\theta}}}(z)^{\frac{1}{\theta}}$, $dx=\frac{p}{\theta\lambda^{\frac{1}{\theta}}} z^{\frac{1}{\theta}-1}\, dz$

And the second integration $u= \lambda\left(\frac{x}{p}\right)^\theta (1 + C^\theta)$, $x=\frac{p}{\lambda^{\frac{1}{\theta}}(1+C^\theta)^{\frac{1}{\theta}}}(u)^{\frac{1}{\theta}}$, $dx=\frac{p}{\theta\lambda^{\frac{1}{\theta}}(1+C^\theta)^{\frac{1}{\theta}}} u^{\frac{1}{\theta}-1}\, dz$

$$W_D = p^\theta\lambda^{-\frac{1}{\theta}}\,\Gamma\left(\frac{1}{\theta}+1\right) - p^\theta\lambda^{-\frac{1}{\theta}}\,\Gamma\left(\frac{1}{\theta}+1\right)[1 - \frac{1}{(1+c^\theta)^{\frac{1}{\theta}+1}}]$$

$$=p^\theta\lambda^{-\frac{1}{\theta}}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\epsilon}\right]$$

Where $\in$ is $\left(1 + c^\theta\right)^{\frac{1}{\theta}+1}$

Then the pdf of Double Weighted Exponential Pareto Distribution (DWEPD) is:

$$g(x,\theta,\lambda,p,c)= \frac{\frac{\theta\lambda}{p}\; x^\theta\, e^{-\lambda(\frac{x}{p})^\theta}[1-e^{-\lambda(\frac{cx}{p})^\theta}]}{p^\theta\lambda^{-\frac{1}{\theta}}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\epsilon}\right]}$$

$$g(x,\theta,\lambda,p,c)= \frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\epsilon}\right]}\; x^\theta\, e^{-\lambda(\frac{x}{p})^\theta}\left[1 - e^{-\lambda(\frac{cx}{p})^\theta}\right], x \geq o\, , c, \theta, \lambda > 0 \qquad \ldots(2)$$

where $K= \frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\epsilon}\right]}$

$$g(x,\theta,\lambda,p,c)=kx^\theta y\left(1 - C^\theta y\right) \qquad \ldots(3)$$

where y=$\mathbf{e}^{-\lambda(\frac{x}{p})^{\theta}}$
and its Cumulative distribution function CDF is given by
G(x)= $\int_0^x g(x)dx$

G(x,θ,λ,p,c)=$\int_0^x \frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\epsilon}\right]}\quad x^{\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]dx$

Let k= $\frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\epsilon}\right]}$

G(x,θ,λ,p,c) = k $\int_0^x\ x^{\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\,[1-e^{-\lambda(\frac{cx}{p})^{\theta}}]dx$

Let the first integration is:

Let u= $\lambda(\frac{x}{p})^{\theta}$ x=$\frac{pu^{\frac{1}{\theta}}}{\lambda^{\frac{1}{\theta}}}$ dx=$\frac{pu^{\frac{1}{\theta}-1}}{\theta\lambda^{\frac{1}{\theta}}}\ du$

And let the second integration is:

Let z= $\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})$ x= $\frac{pu^{\frac{1}{\theta}}}{\lambda^{\frac{1}{\theta}}(1+c^{\theta})^{\frac{1}{\theta}}}$ dx=$\frac{pu^{\frac{1}{\theta}-1}}{\theta\lambda^{\frac{1}{\theta}}(1+c^{\theta})^{\frac{1}{\theta}}}dz$

$=k\,[\int_0^{\lambda(\frac{t}{p})^{\theta}}\ x^{\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\,dx-\int_0^{\lambda\left(\frac{t}{p}\right)^{\theta}(1+C^{\theta})}\ x^{\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\,e^{-\lambda(\frac{cx}{p})^{\theta}}dx]$

G(x,θ,λ,p,c)=$\frac{1}{\Gamma\left(\frac{1}{\theta}+1\right)[\epsilon-1]}\left[\gamma(\frac{1}{\theta}+1,\lambda(\frac{x}{p})^{\theta}\right]$- $\frac{1}{\Gamma\left(\frac{1}{\theta}+1\right)\epsilon[\epsilon-1]}\left[\gamma(\frac{1}{\theta}+1,\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})\right]$

G(x,θ,λ,p,c)=$\frac{1}{\Gamma\left(\frac{1}{\theta}+1\right)[\epsilon-1]}\left[\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}]-\frac{1}{\epsilon}\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}\left(1+C^{\theta}\right)]\right]$ ... (4)

2. The Statistical properties of DWEPD

Statistical properties of DWEPD throughout computing the mean, variance, and standard deviation, coefficient of variation, harmonic mean, and moments presented as follow:

2.1. Moments of DWEPD

The rth moment of DWEPD can be calculated as:

$$E(X^r)=\int_0^{\infty} X^r g(x,\theta,p,\lambda.C)dx$$

$$E(X^r)=k\int_0^{\infty} X^r\ \ x^{\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]dx$$

$$=\int_0^{\infty}\ x^{r+\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]dx$$

$$=\int_0^{\infty}\ x^{r+\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}dx-\int_0^{\infty}\ x^{r+\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}-\lambda(\frac{cx^{\theta}}{p)}}\,dx$$

$$=(\frac{p}{\lambda^{\frac{1}{\theta}}})^r\frac{1}{(1-\frac{1}{\epsilon})}\frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}-(\frac{p}{\lambda^{\frac{1}{\theta}}})^r\frac{1}{(1-\frac{1}{\epsilon})}\frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}\frac{1}{(1+c^{\theta})^{\frac{r}{\theta}}}$$

$$E(X^r)=(\frac{p}{\lambda^{\frac{1}{\theta}}})^r\frac{1}{(1-\frac{1}{\epsilon})}\frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}\left[1-\frac{1}{(1+c^{\theta})^{\frac{r}{\theta}}}\right]$$

Let $\epsilon=\left(1+c^{\theta}\right)^{\frac{1}{\theta}+1}$ and $\epsilon_r=(1+c^{\theta})^{\frac{r}{\theta}}$, $r=1,2,3\ldots\ldots$

$$E(X^r) = \left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^r \frac{1}{(1-\frac{1}{\in})} \frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\in_r}\right] \quad \dots (5)$$

Now, the mean can be obtained, as well as variance, standard deviation, coefficient of variation form eq (5) as follows:

Mean

To find mean put r=1 given by

$$E(x) = \mu = \left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right) \frac{1}{\left(1-\frac{1}{\in}\right)} \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\Gamma\left(\frac{1}{\theta}+1\right)} \left[1-\frac{1}{\in_1}\right] \quad \dots (6)$$

Where $\in_1 = \left(1+c^{\theta}\right)^{\frac{1}{\theta}}$

Variance

$$\sigma^2 = E(X^2) - \left(E(x)\right)^2$$

$$\sigma^2 = \left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^2 \frac{1}{(1-\frac{1}{\in})} \frac{\Gamma(\frac{3}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\in_2}\right] - \left[\left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right) \frac{1}{\left(1-\frac{1}{\in}\right)} \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\Gamma\left(\frac{1}{\theta}+1\right)} \left[1-\frac{1}{\in_1}\right]\right]^2 \quad \dots (7)$$

Where $\in_2 = (1+c^{\theta})^{\frac{2}{\theta}}$

Standard deviation

$$\sigma = \sqrt{\left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^2 \frac{1}{(1-\frac{1}{\in})} \frac{\Gamma(\frac{3}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\in_2}\right] - \left[\left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right) \frac{1}{\left(1-\frac{1}{\in}\right)} \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\Gamma\left(\frac{1}{\theta}+1\right)} \left[1-\frac{1}{\in_1}\right]\right]^2} \quad \dots (8)$$

Coefficient of variation

$$C.V = \frac{\sigma}{\mu} = \frac{\sqrt{\left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^2 \frac{1}{(1-\frac{1}{\in})} \frac{\Gamma(\frac{3}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\in_2}\right] - \left[\left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right) \frac{1}{\left(1-\frac{1}{\in}\right)} \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\Gamma\left(\frac{1}{\theta}+1\right)} \left[1-\frac{1}{\in_1}\right]\right]^2}}{\left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right) \frac{1}{\left(1-\frac{1}{\in}\right)} \frac{\Gamma\left(\frac{2}{\theta}+1\right)}{\Gamma\left(\frac{1}{\theta}+1\right)} \left[1-\frac{1}{\in_1}\right]} \quad \dots (9)$$

Moment Generation Function:

The moment generating function of DWEPD is given by

$$M_x(t) = \int_0^{\infty} e^{-tx} g_w(x,\theta,\lambda,p,c)dx$$

$$= \int_0^{\infty} (1+tx+\frac{(tx)^2}{2!}\dots) g_w(x,\theta,\lambda,p,c)dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(x^r)$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^r \frac{1}{(1-\frac{1}{\in})} \frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\in_r}\right] \quad \dots (10)$$

3. Reliability Analysis

3.1. Reliability function $R(x)$.

The reliability function or, known as survival function $R(x)$ can be derived by using the cumulative distribution function (c.d.f) as follows

$$R(x) = 1 - G_w(x)$$
$$=1 - \frac{1}{\Gamma\left(\frac{1}{\theta}+1\right)[\in -1]}\left[\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}] - \frac{1}{\in}\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})]\right]$$

$$=\frac{\Gamma\left(\frac{1}{\theta}+1\right)[\in -1] - \left[\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}] - \frac{1}{\in}\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})]\right]}{\Gamma\left(\frac{1}{\theta}+1\right)[\in -1]} \quad \ldots(11)$$

3.2. Hazard Function $H(x)$

$H(x)$ denotes the instantaneous rate function or (the Hazard function). Given that the unit has survived until x, the hazard function of x, provided that the conditional probability density of failure at time x or interpreted as instantaneous rate. We can define the Hazard function as

$$H(x) = \frac{g_W(x)}{R(x)}$$

$$H(x) = \frac{\frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\in}\right]}\; x^{\theta}\, e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]}{\frac{\Gamma\left(\frac{1}{\theta}+1\right)\in\;[\in -1]\; - \;\left[\in\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}] - \gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})]\right]}{\Gamma\left(\frac{1}{\theta}+1\right)\in[\in -1]}} \quad \ldots(12)$$

3.3. Reverse Hazard function $\emptyset(x)$

The best describes to reverse hazard function is that you can determine it by the approximate probability of disappointment or failure in $[x\,, x+dx]$. Considering that the loss occurred or failure in $[0, X]$. The function of reverse hazard $\emptyset(x)$ is defined to be

$$\emptyset(x) = \frac{g_W(x)}{G_W(x)}$$

$$\emptyset(x) = \frac{\frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\in}\right]}\; x^{\theta}\, e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]}{\frac{1}{\Gamma\left(\frac{1}{\theta}+1\right)[\in -1]}\left[\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}] - \frac{1}{\in}\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})]\right]}$$

$$\emptyset(x) = \frac{\theta\lambda^{\frac{1}{\theta}+1}\;\in}{p^{\theta+1}}\frac{x^{\theta}\, e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]}{\left[\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}] - \frac{1}{\in}\gamma[(\frac{1}{\theta}+1),\lambda(\frac{x}{p})^{\theta}(1+C^{\theta})]\right]} \quad \ldots(13)$$

4. Estimation methods

As we refer above, this work introduced two estimation methods for the four parameters $(\theta\,, p\,, \lambda\,, C)$. The outcomes of the simulation procedure explained, but after giving some details about the estimators.

4.1. Maximum likelihood method

IF $X_1, X_2, \ldots\ldots, X_n$ are a $r.s.'s$ from DWEP distribution, then the Likelihood function is:

$$Lg(x,\theta,\lambda,p,c)=\prod_{i=1}^{n}\left[\frac{\theta\lambda^{\frac{1}{\theta}+1}}{p^{\theta+1}\,\Gamma\left(\frac{1}{\theta}+1\right)\left[1-\frac{1}{\in}\right]}\quad x^{\theta}\,e^{-\lambda(\frac{x}{p})^{\theta}}\left[1-e^{-\lambda(\frac{cx}{p})^{\theta}}\right]\right]$$

$Lng(x,\theta,\lambda,p,c)$

$$=Ln\theta+\left(\frac{1}{\theta}+1\right)Ln\lambda-(\theta+1)LnP-Ln\Gamma\left(\frac{1}{\theta}+1\right)-Ln\left(1-\frac{1}{(1+C^{\theta})^{\frac{1}{\theta}+1}}\right)+\theta LnX$$

$$-\lambda(\frac{X}{p})^{\theta}+Ln(1-e^{-\lambda\left(\frac{Cx}{p}\right)^{\theta}}\qquad ...(14)$$

$$\frac{\partial Lng(x)}{\partial\theta}=\frac{1}{\theta}-\frac{Ln\lambda}{\theta^{2}}-LnP-\frac{\bar{\Gamma}\left(\frac{1}{\theta}+1\right)}{\Gamma\left(\frac{1}{\theta}+1\right)}+\frac{\theta\left(\frac{1}{\theta}+1\right)C^{\theta-1}(1+C^{\theta})^{\frac{1}{\theta}}}{1-(1+C^{\theta})^{\frac{1}{\theta}+1}}+LnX-\lambda\theta(\frac{x}{p})^{\theta-1}$$

$$-\frac{\lambda\theta(\frac{Cx}{p})^{\theta-1}e^{-}\lambda(\frac{Cx}{p})^{\theta}}{1-e^{-}\lambda(\frac{Cx}{p})^{\theta}}\qquad ...(15)$$

$$\frac{\partial Lng(x)}{\partial p}=\frac{\theta\lambda}{\theta p}(\frac{x}{p})^{\theta}-\frac{(\theta+1)}{p}-\frac{\frac{\lambda\theta}{p}(\frac{Cx}{p})^{\theta}e^{-}\lambda(\frac{Cx}{p})^{\theta}}{1-e^{-}\lambda(\frac{Cx}{p})^{\theta}}\qquad ...(\mathbf{16})$$

$$\frac{\partial Lng(x)}{\partial\lambda}=\frac{\frac{1}{\theta}+1}{\lambda}-(\frac{x}{p})^{\theta}-\frac{(\frac{Cx}{p})^{\theta}e^{-}\lambda(\frac{Cx}{p})^{\theta}}{1-e^{-}\lambda(\frac{Cx}{p})^{\theta}}\qquad ...(\mathbf{17})$$

The numerical solution can be used to determined eq. (17) instantaneously, since this equation can be equal to zero. Therefore, we obtain $\theta^{\wedge}_{MLE}$ and $p^{\wedge}_{MLE}$ $\lambda^{\wedge}_{MLE}$ $C^{\wedge}_{MLE}$ as M. L. E. estimators of θ, p, λ, C respectively.

4.2. Method of moment

An independent random sample $r.s.$, $X_1, X_2, \dots \dots, X_n$ from the DWRD with parameters θ, p, λ, and C. The moment estimation method is obtained by measuring population moments and equating them with sample moments.

$$m_r=\frac{\sum_{i=1}^{n}{x_i}^{r}}{n}\qquad ...(18)$$

$$\mu_r=E(x^{r})\qquad ...(19)$$

$$m_r=\mu_r$$

$$\boldsymbol{E}(X^{r})=(\frac{p}{\lambda^{\frac{1}{\theta}}})^{r}\frac{1}{(1-\frac{1}{\in})}\frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}\left[1-\frac{1}{(1+c^{\theta})^{\frac{r}{\theta}}}\right]$$

Let $\in=\left(1+c^{\theta}\right)^{\frac{1}{\theta}+1}$ and $\in_r=(1+c^{\theta})^{\frac{r}{\theta}}$ $, r=1,2,3 \dots ....$

$$E(X^{r})=(\frac{p}{\lambda^{\frac{1}{\theta}}})^{r}\frac{1}{(1-\frac{1}{\in})}\frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}\left[1-\frac{1}{\in_r}\right]$$

$$\frac{\sum_{i}^{n}x_i}{n}=\bar{X}=(\frac{\mathrm{p}}{\lambda^{\frac{1}{\theta}}})^{r}\frac{1}{(1-\frac{1}{\in})}\frac{\Gamma(\frac{r+1}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}\left[1-\frac{1}{\in_r}\right]\qquad ...\ (20)$$

$$\frac{\sum x_i^{2}}{n}=(\frac{p}{\lambda^{\frac{1}{\theta}}})^{2}\frac{1}{(1-\frac{1}{\in})}\frac{\Gamma(\frac{3}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)}\left[1-\frac{1}{\in_2}\right]\qquad ...\ (21)$$

$$\frac{\sum x_i^3}{n} = \left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^3 \frac{1}{(1-\frac{1}{\epsilon})} \frac{\Gamma(\frac{4}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\epsilon_3}\right] \quad \ldots \; (22)$$

$$\frac{\sum x_i^4}{n} = \left(\frac{p}{\lambda^{\frac{1}{\theta}}}\right)^4 \frac{1}{(1-\frac{1}{\epsilon})} \frac{\Gamma(\frac{5}{\theta}+1)}{\Gamma(\frac{1}{\theta}+1)} \left[1-\frac{1}{\epsilon_4}\right] \quad \ldots \; (23)$$

By solving the four equations (20), (21), (22), and (23) simultaneously (numerical method), get $(\hat{\theta}_{mom}, \hat{p}_{mom}, \hat{\lambda}_{mom}, \hat{c}_{mom})$ as an estimate of θ, p, ,λ and c respectively.

4.3. Percentiles estimation (PE)

Initially, Kao (1959) discovered this technique by graphically approximating the best linear unbiased estimators. By fitting a straight line to the theoretical points calculated from the distribution function, and the sample percentile points, the estimators could be found. In the case of a DWEP distribution, because of the nature of its distribution function, it is likely to use the same idea to evaluate the estimators of θ, p, λ, and C based on PE. Since G(x) is separated by Al-khadim in (2014). Firstly, determined numerically the value of x where x= G-1(x, θ, P, λ, C), since P_i is the estimate of G $(X_{(i)}, \theta, P, \lambda)$ $\hat{\theta}_{PE}$, $\hat{P}_{PE}$, $\hat{\lambda}_{PE}$, $\hat{C}_{PE}$ can be determined by minimizing

$\sum_{i=1}^{n}\left[x_{(i)} - G^{-1}(P_i, \theta, P, \lambda, C)\right]^2$ Concerning θ, P, λ, C where

$E\left[G(x_{(i)}\right] = P_i = \frac{i}{n+1}$ is the most used estimator of G(X(i)).

4.4. Numerical study

The Monte-Carlo simulation study was conducted by using MATLAB to test the ability of the estimation methods presented in paragraph (5) which is the Maximum Likelihood method, the moment's method, and the Percentile method. Assumed a variety of theoretical parameters for DWEPD which is $\boldsymbol{\theta = 2.5}, 3, \boldsymbol{5}$ $\boldsymbol{\lambda = 2.5}, 3$ $\boldsymbol{P = 2.5, 2, 1.5}$ $\boldsymbol{C = 2.}5,2,1.5$, and sample sizes (25, 50,100,150) and the replication (1000) for each simulation experiment to obtain the homogeneity of the results. The results were compared by using MSE and MAPE. The simulation study showed the preference of the Percentiles method over the other methods at all the size of samples, 50,100, and the method of the Maximum Likelihood method at the size of 150.

Table 1. Results of simulation under all sample sizes and theoretical parameters

$\boldsymbol{\theta = 2.5}$ $\boldsymbol{\lambda = 2.5}$ $\boldsymbol{P = 2.5}$ $\boldsymbol{C = 2.5}$						
n	Method		$\hat{\theta}$	$\hat{\lambda}$	$\hat{p}$	$\hat{c}$
25	ML.	Parameter	2.43116	2.72011	1.99819	2.299821
		MSE	0.033442	0.043528	0.03328	0.023310
		MAPE	0.120095	0.334486	0.22555	0.129892
	Mom.	Parameter	2.315585	2.182231	2.097783	2.6445863
		MSE	0.025638	0.122396	0.674627	0.014844
		MAPE	0.044316	0.230219	0.409433	0.041808
	Prec.	Parameter	2.499765	2.519132	2.497654	2.499981
		MSE	0.004383	0.004733	0.033567	0.003943
		MAPE	0.023644	0.022435	0.063439	0.020098
Best			Prec.	Prec.	Prec.	Prec.
50	ML.	Parameter	2.132167	2.321456	1.567321	2.312222
		MSE	0.154042	0.095019	0.065498	0.159717
		MAPE	0.116574	0.170389	0.102865	0.137322
	Mom.	Parameter	2.422089	1.186318	1.843517	2.448652
		MSE	0.019395	0.120794	0.666337	0.013555
		MAPE	0.038842	0.229056	0.407341	0.040217
	Prec.	Parameter	2.504587	2.490333	2.550866	2.501567

		MSE	0.003511	0.002911	0.005655	0.001335
		MAPE	0.026894	0.032228	0.034321	0.028889
Best			Prec.	Prec.	Prec.	Prec.
100	ML.	Parameter	2.238777	2.468719	1.901123	2.248716
		MSE	0.118367	0.09894	0.053495	0.16713
		MAPE	0.118729	0.18177	0.106784	0.242325
	Mom.	Parameter	2.411089	1.185317	1.843578	2.488652
		MSE	0.018678	0.122151	0.670825	0.056282
		MAPE	0.036836	0.231208	0.408862	0.03897
	Prec.	Parameter	2.503439	2.506796	2.501521	2.50167
		MSE	0.00345	0.003476	0.003487	0.00334
		MAPE	0.01943	0.034451	0.026689	0.02098
Best			Prec.	Prec.	Prec.	Prec.
150	ML.	Parameter	2.507781	2.32119	2.290115	2.35671
		MSE	0.002156	0.004016	0.004545	0.004455
		MAPE	0.019604	0.031698	0.024932	0.019788
	Mom.	Parameter	2.353477	2.278122	1.998738	2.459919
		MSE	0.008964	0.081852	0.576677	0.032507
		MAPE	0.022787	0.189866	0.411746	0.057812
	Prec.	Parameter	2.456565	1.949285	1.453535	2.446147
		MSE	0.003758	0.002913	0.003422	0.002743
		MAPE	0.021943	0.030976	0.025358	0.01753
Best			ML.	Prec.	Prec.	Prec.

Table 2. Results of simulation under all sample sizes and theoretical parameters

$\boldsymbol{\theta = 2.5}$ $\boldsymbol{\lambda = 2.5}$ $\boldsymbol{P = 2}$ $\boldsymbol{C = 2}$						
n	Method		$\hat{\boldsymbol{\theta}}$	$\hat{\boldsymbol{\lambda}}$	$\hat{\boldsymbol{p}}$	$\hat{\boldsymbol{c}}$
25	ML.	Parameter	1.989226	1.76112	1.434579	1.630657
		MSE	0.155568	0.083361	0.048788	0.164314
		MAPE	0.139612	0.179027	0.094444	0.193864
	Mom.	Parameter	2.39042	2.070324	1.878362	1.933908
		MSE	0.88506	0.003422	0.013248	2.182283
		MAPE	0.376688	0.035058	0.039678	0.75479
	Prec.	Parameter	2.000176	2.145712	1.502755	2.500455
		MSE	0.003323	0.003438	0.016752	0.003351
		MAPE	0.019751	0.033887	0.060636	0.021842
Best			Prec.	Mom	Mom.	Prec.
50	ML.	Parameter	1.966672	1.679849	1.319831	1.545619
		MSE	0.125537	0.081299	0.055723	0.158464
		MAPE	0.124465	0.166712	0.102588	0.17176
	Mom.	Parameter	1.774784	1.478322	1.867458	1.623333
		MSE	0.0087899	0.129132	0.727143	0.025344
		MAPE	0.030166	0.236965	0.426351	0.075425
	Prec.	Parameter	2.48678	2.399983	2.499432	2.468535
		MSE	0.003353	0.003315	0.006853	0.003111
		MAPE	0.020124	0.033493	0.033668	0.025431
Best			Prec.	Prec.	Prec.	Prec.
100	ML.	Parameter	1.689222	1.893357	1.278898	1.61867
		MSE	0.132843	0.086648	0.056059	0.16232
		MAPE	0.127653	0.169737	0.103478	0.172912
	Mom.	Parameter	1.884293	1.158242	1.819322	2.563181
		MSE	0.008671	0.131741	0.726526	0.025215

		MAPE	0.029974	0.239542	0.425972	0.073373
	Prec.	Parameter	2.500178	2.402288	1.981532	2.00987
		MSE	0.003377	0.00297	0.002922	0.003119
		MAPE	0.020228	0.02959	0.017556	0.022521
Best			Prec.	Prec.	Prec.	Prec.
150	ML.	Parameter	2.328891	2.245921	1.968712	2.198883
		MSE	0.002818	0.002913	0.003182	0.002891
		MAPE	0.068021	0.028824	0.025361	0.024973
	Mom.	Parameter	2.502003	1.885341	1.660188	2.515821
		MSE	0.000891	0.090837	0.747855	0.036425
		MAPE	0.0148681	0.200133	0.432276	0.094165
	Prec.	Parameter	2.457882	2.493444	1.444914	2.450834
		MSE	0.003264	0.003773	0.003057	0.00204
		MAPE	0.019688	0.036724	0.02367	0.023819
Best			Mom.	ML.	Prec.	Prec.

Table 3. Results of simulation under all sample sizes and theoretical parameters

$\theta = 3$ $\lambda = 3$ $P = 1.5$ $C = 1.5$						
n	Method		$\hat{\theta}$	$\hat{\lambda}$	$\hat{p}$	$\hat{c}$
25	ML.	Parameter	2.677578	2.21145	1.13332	1.308991
		MSE	0.126176	0.082487	0.054755	0.136227
		MAPE	0.123127	0.164677	0.080667	0.156845
	Mom.	Parameter	1.997611	2.195556	2.018333	1.820322
		MSE	0.018282	0.260266	1.011292	0.062223
		MAPE	0.048435	0.338282	0.301926	0.134415
	Prec.	Parameter	2.95979	3.080061	1.545561	1. 49112
		MSE	0.003311	0.003165	0.007789	0.003098
		MAPE	0.020167	0.021799	0.03001	0.024377
Best			Prec.	Prec.	Prec.	Prec.
50	ML.	Parameter	2.578911	2.396541	1.241333	1.189921
		MSE	0.127152	0.072488	0.055756	0.137125
		MAPE	0.123323	0.164693	0.080732	0.156816
	Mom.	Parameter	2.745666	1.999653	1.917333	2.001125
		MSE	0.018235	0.260667	1.011234	0.062221
		MAPE	0.048433	0.368831	0.411956	0.121153
	Prec.	Parameter	3.26822	3.003373	1.500855	2.499216
		MSE	0.003219	0.003255	0.00845	0.003161
		MAPE	0.020188	0.031765	0.03127	0.024245
Best			Prec.	Prec.	Prec.	Prec.
100	ML.	Parameter	1.677211	2.301118	1.341111	2.145919
		MSE	0.118736	0.0880924	0.051474	0.150318
		MAPE	0.118812	0.16923	0.078055	0.165094
	Mom.	Parameter	1.760776	1.49383	2.007874	2.62093
		MSE	0.018388	0.260308	1.011934	0.060743
		MAPE	0.048477	0.338453	0.4012	0.119912
	Prec.	Parameter	2.967881	3.111176	1.509011	1.49001
		MSE	0.003513	0.003441	0.00346	0.003234
		MAPE	0.020176	0.032902	0.019842	0.02355
Best			Prec.	Prec.	Prec.	Prec.
150	ML.	Parameter	1.948334	2.114566	1.551211	2.455901
		MSE	0.002642	0.003295	0.003281	0.002799

		MAPE	0.018841	0.032528	0.018668	0.025637
	Mom.	Parameter	1.713444	1.488052	1.947384	2.532011
		MSE	0.007725	0.20098	1.026657	0.082688
		MAPE	0.032217	0.298254	0.405172	0.142688
	Prec.	Parameter	1.945555	2.450336	1.450625	2.448172
		MSE	0.004477	0.003077	0.003351	0.00379
		MAPE	0.020732	0.029911	0.019865	0.027223
Best						

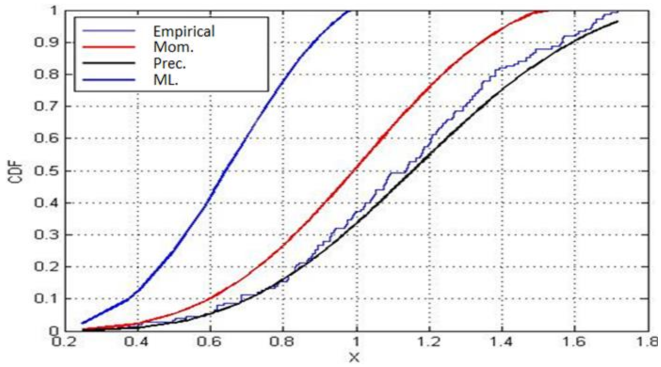

Figure 1. The Cumulative distribution function under $\theta = 2.5 \quad \lambda = 2.5 \quad P = 2.5 \;\; C = 2.5$.

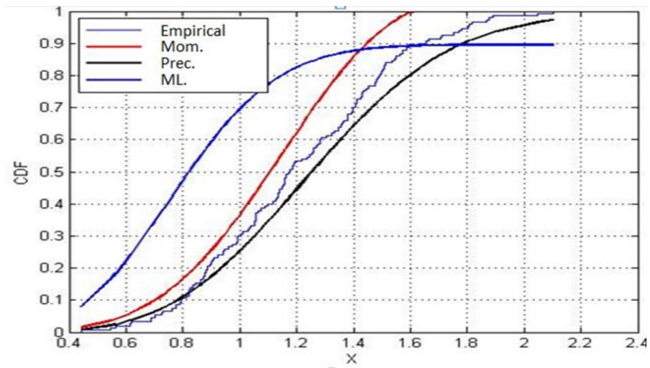

Figure 2. The Cumulative distribution function under $\theta = 2.5 \quad \lambda = 2.5 \quad P = 2 \quad C = 2$

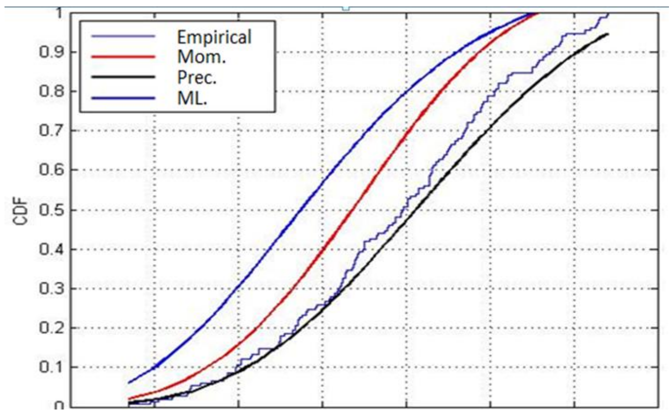

Figure 3. The Cumulative distribution function under $\theta = 3 \quad \lambda = 3 \quad P = 1.5 \quad C = 1.5$

Applied Side:

The data were collected and applied to the best methods used in the research, which represents the period of survival of the patient until death for patients with breast cancer. Medical in the holy province of Karbala (100). After each patient took doses of chemotherapy from the chemotherapy unit, the times until death occurred in months for the period (2016-2018) and the following table shows the real data under investigation:

Table 4. Real Data sheet

11.7	10.7	10.7	10.5	12.4	12.4	11.3	11.3	10.3	10.2
11.5	11.3	11.3	11.3	11.3	11.2	11.1	11.1	11.9	10.9
11.9	11.9	11.8	11.8	11.8	11.6	11.6	11.5	17.5	21.5
12.4	12.5	12.5	21.5	12.3	12.1	12	12	11.9	11.9
13.28	13.25	13.22	13.1	13	13	12.9	12.8	12.8	12.8
13.9	14.9	13.7	13.6	13.6	13.5	13.4	15.4	11.3	11.3
14.7	14.6	14.6	15.5	14.4	14.4	14.3	14.27	14.26	14
15.6	14.6	16.4	15.3	15.3	15.3	15.13	14.9	14.8	14.7
17.8	17.7	17.7	17.6	17.5	17.3	16.7	17.6	16.3	19.1
23.6	21.15	18.5	18.5	19.2	19.1	18.7	18.6	18.5	17.7

For fitting data on the survival period until death according to DWEPD of the four parameters. A goodness of fit test was conducted which includes four types of tests to analyze the real data sample in estimating the parameters by Percentiles method and its application to the real experience data of breast cancer diseases, which is best estimated through experimental simulation, table (4) shows the parameter estimates for the proposed distribution (DWEPD) for the goodness of fit(Chi-Squared, Anderson–Darling, Kolmogorov-Smirnov, Cramer Van Mises) and the result as following:

Table 5. Results of the goodness of fit for real Data

Model	Parameter				Chi-square	A_d^*	K. S	W_d^*
	θ	λ	p	c				
DWEPD	2.6888	2.322	2.3111	2.517	0.6811	2.0999	0.09554	0.0059
Exp-Pareto	2.8451	3.7778	3.3114		432333	7.6755	0.22212	0.0151
Pareto	3.6719		3.2187		77.8755		0.65543	0.053
Exponential		.7814			79.8776	21.433	0.36754	0.0755

From table (2) we note that the bias parameter (C) we observe its value (2.517) based on default values on the simulation side. The values of the calculated parameters were compatible with the default values for the parameters shown on the simulation side. And to test the hypothesis (H_0 : X∼DWEP against H_1: X ≁ DWEP) the table shows that the results of the H_0 null hypothesis test show, according to the four criteria, the acceptance of this hypothesis a significant level of 0.05)), i.e., the real data follow the proposed distribution (DWEP), where we have been confirmed by comparing the four tests while Chi-square statistic for the distribution of Exponential- Pareto value and the distribution of Pareto, Exponential, and this indicator of the values of the three alternative distributions (for the proposed distribution) confirms the rejection of the null hypothesis.

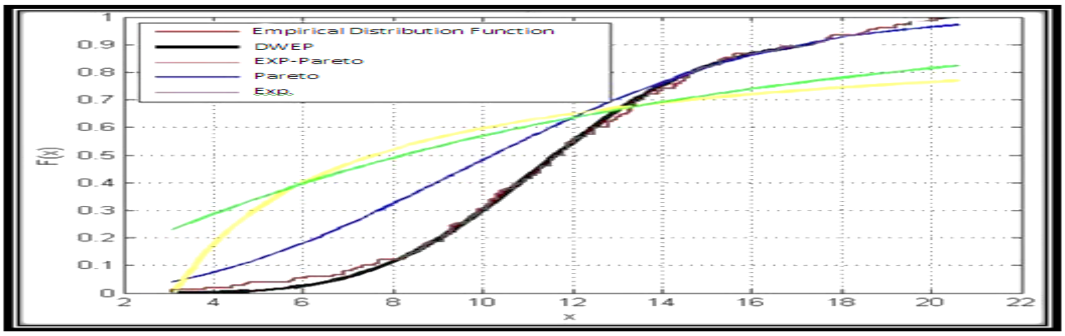

Figure 4. The cumulative distribution function for real data

This figure showed clear approximate among empirical curve for double-weighted exponential Pareto and real data curve which it refers to the accuracy of fitting data according to four tests for fitting.

5. Conclusions

1. The researcher input double parameter of bias (C) for the distribution of (Exponential-Pareto), so the distribution was called distribution ((Double Weighted Exponential-Pareto) and was proved to be a weighted probability distribution.
2. It was found that the best value of the bias parameter C contributes to the elimination of the bias is when (C> 2.5) results from the adoption of data of different sizes, and this reflects the importance of weighted probability distributions rather than probability distributions only if the researcher is studying or interested in the analysis of data originally from different samples sizes especially in health and life applications.
3. The best method of estimation of parameters was the Percentiles method at all the size of samples, 50,100 because have less MSE and MAPE.
4. The priority of the Maximum Likelihood method at the size of 150.
5. When increasing the sample size, then the Percentiles method and maximum Likelihood method is the best.

Recommendations

1- Extend the research to include other weighted vehicle distributions, as this is important in estimating operating times or failures and in evaluating expensive medical trials.
2- Dependence on other indicators to reduce or reduce uncertainty as well as the Renyi Entropy scale such as Shannon- Entropy, and others.
3- Addition of other methods of estimation, other than those adopted by the researcher such as Bayesian methods. The research on the proposed model can be expanded and converted into complex probability distributions to accommodate double data.

Acknowledgments

The authors would like to thank the editor and the referees for their careful reading and comments.

References

[1] W. Bodhisuwan, N. Nanuwong, and C. Pudprommarat, "Parameter Estimation for the Length Biased Beta-Pareto Distribution and Application", *Walailak J Sci & Tech*, vol.13, no.5, pp. 301 – 315, 2016.
[2] R. Kan, T. Kaosol, P. Tekasakul, and S. Tekasakul, "Risk Assessment of Emitted Particle-bound Polycyclic Aromatic Hydrocarbons from Lignite-biomass Pelletization Burning: Size Distribution and Human Health Effects", *Walailak J Sci & Tech*, vol.16, no.10, pp. 805-816, 2019.
[3] S. Aryuyuen, and W. Bodhisuwan, "The Truncated Power Lomax Distribution: Properties and Applications", *Walailak J Sci & Tech*, vol.16, no.9, pp. 655-668, 2019.
[4] R. Fisher, "The effect of methods of ascertainment upon the estimation of frequencies", *The Annals of Eugenics*, vol.6, pp.13-25, 1934.
[5] C. Rao, *"On discrete distributions arising out of methods of ascertainment, in Classical and Contagious Discrete distributions*", Pergamum Press and Statistical Publishing Society, Calcutta, pp. 320-332, 1965.
[6] H. Ghdhaib, and M. Abd Al-Rahem, "Comparison of four methods to estimate the scale parameter for Extended Poisson Exponential distribution", *IOP Conf. Ser. Mater. Sci. Eng.*, vol.571, p.012029, 2019.
[7] K. Al-khadim, and A. Hantoosh, "Double Weighted distribution, and double-weighted exponential distribution", *Mathematical theory and modeling*, vol.3, pp. 124-134, 2013.
[8] R. Gupta, and S. Kirmani, "The Role of Weighted Distributions in Stochastic Modeling", *Comm. Statis.*, vol.19, pp.3147-31, 1990.
[9] N. Rashwan, "The Double Weighted Rayleigh Distribution: Properties and Estimation", *International Journal of Scientific & Engineering Research*, vol.4, no.12, pp. 1084 – 1089, 2013.
[10] A. Ahmad, S. Ahmad, and A. Ahmed, "Characterization and Estimation of Double Weighted Rayleigh Distribution", *Journal of Agriculture and Life Sciences*, vol.1, no.2, 2014.
[11] K. Al-Kadim, and A. Hantoosh, "Double Weighted Inverse Weibull distribution", *Handbook on the Emerging Trends in Scientific Research*, pp. 386 – 398, 2014.
[12] X. Jing, Weighted Inverse Weibull and Beta- inverse Weibull distributions, Master Dissertation, Statesboro, Georgia, 2010.

[13] F. Merovci, and I. Elbatal, "Weibull Rayleigh distribution theory and applications", *Applied mathematics and information science*, vol.4, pp.2127-2137, 2016.

[14] A. Basheer, "Alpha power inverse Weibull distribution with reliability application", *Journal of Taibah University for Science*, vol.13, no.1, pp 423 – 432, 2016.

[15] A. Saghir, and M. Saleem, "Double Weighted Weibull Distribution Properties and Application. *Mathematical Theory and Modeling"*, vol.6, no.7, pp. 28 – 46, 2016.

[16] B. Oluyede, and E. George, "On Stochastic Inequalities and Comparisons of Reliability Measures for Weighted Distributions", *Mathematical problems in Engineering*, vol.8, pp.1-13, 2000.