Periodicals of Engineering and Natural Sciences ISSN 2303-4521
Vol.7, No.1, January 2019, pp. 20~35
Available online at: http://pen.ius.edu.ba

Comparing symmetric and asymmetric volatility estimates for S&P index prices

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Article Info

Received Dec 13, 2018

Keyword:

Volatility
GJR-GARCH
Conditional Variance
Asymmetry
Stock Index

ABSTRACT

In every parametric formula of pricing a financial instrument, factors used in the calculation generally include the volatility estimate. Volatility measures the likely changes of the price for a specific period of time. The accuracy of estimated price strongly relies on the accuracy of provided expected changes in the market volatility for the period of interest. As opposed to other variables, which are assigned values to financial instrument, volatility is the only estimated one. For that reason, big focus of researchers was and still is on improving the volatility estimate. Initiated are different estimation approaches through last few decades. This paper explains few ARCH models, symmetric and asymmetric, and compares their estimates of daily volatility for the Standard and Poor's Indexes.

1. INTRODUCTION

Volatility estimation is grouped into two methods of: (i) historical volatility – learning from past characteristics of price change, while assuming that they will hold; and (ii) implied volatility – calculating the fitting volatility value based on the market assigned price. This paper will compare two historical volatility estimates for Standard and Poor's index, and finally compare them to the realized volatility calculated based on intraday prices.

Initiated by [1] the ARCH model was further developed in various directions. A detailed list of such models, all originating from the Engle's ARCH, is provided by [2]. This paper will mention just few of them, including GARCH (1,1), ARCH-M, and GJR-GARCH. The symmetric model of GARCH(1,1) and asymmetric GJR-GARCH will be applied and compared using ten years of the daily closing price market data of both Standard and Poor's 100 and Standard and Poor's 500 Indexes.

Previoulsy done reviews of ARCH-type models include [3], [2], [4], [5], [6] and [7]. There were vast comparisons of different volatility forecasts and their performance. Some conclude the preference was based on markets as well as on financial instruments (stocks, indexes, bonds, or currency). A detailed review of such kind is [8], which shows that 17 out of 39 preceding studies preferred GARCH (1,1) while remaining 22 studies find the historical volatility (including random walk, historical averages of squared returns, and absolute returns) to perform better.

Similarly, as studies cover different markets, different conclusions of preferable methods for estimation were obtained. Selected EWMA by [9] for Japan, just as [10] did for Singapore. On the other side, [11] find GJR (1,1) to be the choice for Australia where EWMA performed weakly. [12] did a very detailed study, using a continuously compounded daily return of fourteen countries for estimating corresponding volatility both within a week and within a month. Comparing eleven different methods, they found the best performance to be by a weighted moving average model with weight declining by 10% and looking back for 12 periods.

Reached are different conclusions in selecting when estimating the volatility forecast of currency exchange rate. Evaluating the methods on five different currencies, [13] conclude that GARCH(1,1) outperforms others. [14] study the forecasting performance also of the currency exchange rate between the American dollar and currencies of each of Canada, France, Germany, Japan, and U.K. using the market data from 1973 to 1989. While studying the forecast for twelve-weeks and twenty-four-weeks the results were inconclusive, and only a one-week prediction gave an advantage to the GARCH model.

2. METHODOLOGY

The paper starts by explaining the model of GARCH (1,1) in details. Starting from its mathematical logic related to the already learnt behavior of financial returns, followed by its practical application with use of maximum likelihood methods. Similar, but less detailed steps, follow for the GJR-GARCH model.

Knowing that returns do not fit the Gaussian distribution, the GJR model is estimated under assumption of both the Gaussian as well as the T distribution.

By considering the closing prices of that financial instrument, at the end of the days t-1 and t, to be p_{t-1} and p_t, respectively. Accordingly, the value of r_t representing the continuous interest rate for that day 't' would be calculated as:

$$r_t = \ln \frac{p_t}{p_{t-1}}. \tag{1}$$

Based on it all the other calculations will be done. In other words, the information used for calculating the volatility estimation formula is its history of daily prices. No other market information is used. Finally, the data of daily closing prices of S&P100 and S&P500 indexes daily closing price will be used to show real examples of all these models GJR-GARCH and GARCH.

3. ARCH MODELS

Through modifying the idea of Auto-regression (AR) by adding the power of two, [1] introduced the very popular ARCH(m) model. Further developed was the initial ARCH to tens of other models. Initially, the concept of ARCH (1) will be explained, looking only at the information of a single preceding period. Starting with an assumption of return for a period t, represented by r_t, conditional on preceding returns being normally distributed with a constant mean value of μ and with a time-varying conditional variance h_t, is defined as

$$r_t | r_{t-1}, r_{t-2}, \ldots \sim N(\mu,\ h_t) \tag{2}$$
$$h_t = \omega + \alpha(r_{t-1} - \mu)^2;\ \ \omega > 0, \alpha \geq 0. \tag{3}$$

The corresponding value of its period t, residual e_t is calculated by subtracting the mean value of historical returns μ from the return realized in the period t

$$e_t = r_t - \mu \tag{4}$$

The error of the forecast of the squared residual, here represented by the symbol of v_t, based on the estimate of h_t, can be rephrased as the difference between the realized squared difference between the mean and the conditionally expected squared difference from the mean. It is calculated by using

$$v_t = e_t^2 - E[e_t^2 | r_{t-1}, r_{t-2}, \ldots] \tag{5}$$

$$v_t = e_t^2 - h_t. \tag{6}$$

Replacing the h_t in the (3), by the corresponding $e_t^2 - v_t$ from the formula (6) provides a new shape of the ARCH (1) for estimating the difference as

$$e_t^2 = \omega + \alpha e_{t-1}^2 + v_t. \tag{7}$$

Satisfaction, for the outcomes achieved through ARCH, has been reported by among others [15], [16], and [17]. They have confirmed it is suitable for various financial time-series.

The ARCH was further developed by [18], as they were the first to publish an extension of the ARCH model to be multivariate. [17] state a few advantages of the ARCH as the main grounds for its success. According to them, while managing the clustered errors as much as its nonlinearities, ARCH models are simple and easy to handle.

3.1 GARCH(1,1) MODEL

GARCH (p, q) was the first further development of ARCH (p), by assigning some weight β_i to the corresponding lagged conditional variance of h_{t-i}. The GARCH (1,1) model was introduced separately by [19] and [20]. For a single lag GARCH (1,1) a conditional variance for period t gets calculated as

$$h_t = \omega + \alpha(r_{t-1} - \mu)^2 + \beta h_{t-1}. \tag{8}$$

For the process to be stationary a constraint for the sum of α and β to be less than one is added. Considering the values of returns to be normally distributed, corresponding standardized residual is calculated by:

$$z_t = \frac{r_t - \mu}{\sqrt{h_t}} = \frac{e_t}{\sqrt{h_t}} \tag{9}$$

Where, the distribution of z_t, considering its non-dependence on past returns, would be:

$$z_t | r_{t-1}, r_{t-2}, \ldots \sim N(0,1), \tag{10}$$

Then a GARCH (1,1) under conditional normal distribution, would be defined through:

$$r_t = \mu + z_t\sqrt{h_t}, \tag{11}$$

$$z_t \sim \text{i.i.d.}\, N(0,1), \tag{12}$$

$$h_t = \omega + \alpha(r_{t-1} - \mu)^2 + \beta h_{t-1}. \tag{13}$$

Replacing the r_{t-1} part of (8) with the corresponding equivalent based on (6), provides:

$$h_t = \omega + \alpha(z_{t-1}\sqrt{h_{t-1}})^2 + \beta h_{t-1}, \tag{14}$$

It can be rewritten as:

$$h_t = \omega + (\alpha z_{t-1}^2 + \beta)h_{t-1}. \tag{15}$$

Taking into consideration that $z \sim N(0,1)$, which implies E[z²] =1, the expected value of the conditional variance for the period t is calculated as:

$$E[h_t] = \omega + (\alpha + \beta)E[h_{t-1}]. \tag{16}$$

Given the independence of z_{t-1} from h_{t-1}, since the calculation of h_{t-1} includes the standardized residual value of the preceding period, instead a variable of z_{t-2}, the only remaining precondition is that the conditional variance of h_t has a finite expected value. Taking into consideration the previously stated rule of $\omega > 0$ leaves no other option for (16) to be covariance-stationary except by having $\alpha + \beta < 1$. Such conclusion was first proven by (Bollerslev 1986), through taking the process back in time indefinitely.

From the other aspect, formula of (8) for h_t, having a non-zero value of β, shows its dependence on h_{t-1}. Similarly, rewriting the same formula for h_{t-1}, it would be dependent on h_{t-2}, and such relationships can be seen all the way down until h_1 dependence on h_0. This characteristic confirms the dependence of GARCH (1,1) value of conditional variance h_t, on all the previous values of h_i, $i < t$. Thereby, the covariance stationary GARCH (1,1) is comparable to ARCH (∞).

Forming the corresponding formula for h_{t-1}, and substituting it within that of h_t, provides:

$$h_t = \omega + \beta\omega + \alpha(r_{t-2} - \mu)^2 + \alpha\beta(r_{t-3} - \mu)^2 + \beta^2 h_{t-2}. \tag{17}$$

Further step of adding the corresponding equation of h_{t-2}, which itself includes h_{t-3}, provides:

$$h_t = \omega + \beta\omega + \beta^2\omega + \alpha(r_{t-1} - \mu)^2 + \alpha\beta(r_{t-2} - \mu)^2 + \alpha\beta^2(r_{t-3} - \mu)^2 + \beta^3 h_{t-3} \quad (18)$$

By enduring such steps, going back with the substitution for some m number of periods generates:

$$h_t = \sum_{i=0}^{m-1} \omega\beta^i + \sum_{i=1}^{m} \alpha\beta^{i-1}(r_{t-i} - \mu)^2 + \beta^m h_{t-m} \quad (19)$$

If assuming that the process goes back indefinitely, replacing the value of m by infinity, will enable application of the infinite geometric regression which, knowing that GARCH(1,1) requires that $\beta < 1$, allows the replacement of its total sum as:

$$\sum_{i=0}^{\infty} \omega\beta^i = \frac{\omega}{1-\beta} \quad (20)$$

Such that

$$h_t = \frac{\omega}{1-\beta} + \sum_{i=1}^{\infty} \alpha\beta^{i-1}(r_{t-i} - \mu)^2 + \beta^{\infty} h_0. \quad (21)$$

Similarly, knowing that $\beta < 1$ makes $\beta^{\infty} h_0$ approach zero, the $\beta^{\infty} h_0$ part can be excluded from the (21), accordingly providing:

$$h_t = \frac{\omega}{1-\beta} + \sum_{i=1}^{\infty} \alpha\beta^{i-1}(r_{t-i} - \mu)^2. \quad (22)$$

Stated before is that for GARCH (p, q) both $\alpha \geq 0$ and $\beta \geq 0$, their sum $\alpha + \beta < 1$, and their power $\alpha\beta^{i-1}$ turns to be of a negligible value when put to the power of a reasonably big number. The function includes the weight given to each term. The measure of weight decreases exponentially according to its time distance from the present time. Consequently, the weight that would get assigned to the residual e_{t-k}^2 is $\alpha\beta^{k-1}$. As such, GARCH becomes very much comparable to the Estimated Weighted Moving Average (EWMA) method of volatility estimation.

On the other hand, every h_t has its minimum possible value of $\frac{\omega}{1-\beta}$ that might occur through having either α or β assigned to be zero.

3.1.1 GARCH(1,1) MOMENTS

As already stated, z_t is considered to be i.i.d. At the same time knowing that h_t depends on the past values of z, the unconditional expected return gets calculated as:

$$E[r_t] = \mu + E[e_t] = \mu + E\left[\sqrt{h_t}\right]E[z_t]$$

Since $z_t \sim \text{i.i.d.}\, N(0,1)$, then $E[z_t] = 0$, and $E[r_t] = \mu$. On the other hand, the corresponding variance, not dependent on past returns, known as unconditional variance, gets calculated based on the residual:

$$\text{var}(r_t) = E[{e_t}^2] = E[h_t]E[z_t^2] = E[h_t]$$

Since $E[z_t^2] = 1$, it can be concluded, from (16), that:

$$\text{var}(r_t) = \frac{\omega}{1-\alpha-\beta}. \quad (23)$$

Accordingly, the previous conditional variance formula (6) gets rewritten as:

$$h_t = (1 - \alpha - \beta)\sigma^2 + \alpha(r_{t-1} - \mu)^2 + \beta h_{t-1}. \quad (24)$$

Consequently, the GARCH(1,1) conditional variance of h_t can be summarized as a combination of the squared residual $(r_{t-1} - \mu)^2$, the conditional variance h_{t-1}, from the preceding period (t-1) and respectively weighted by α and β, and the remaining weight of $1 - \alpha - \beta$ is assigned to the variance σ^2.

3.2 ARCH-M MODELS

ARCH-in-mean is merely any ARCH model that includes the conditional volatility of h_t in the process of estimating the daily expected return μ, instead of using a constant μ as is the case in ARCH(p) or GARCH(p, q). [21] introduced one of the suggested examples of ARCH-M:

$$\mu_t = \xi + \lambda\sqrt{h_t}. \tag{25}$$

Defining ξ as a risk-free interest rate, λ represents a weight of the risk parameter of estimated conditional volatility.

Practical use of ARCH-M model, on the interest-rate data, was done by [21], who estimated the time-varying risk premium with outcome showing a good data fit. Tests of the ARCH-M model, on returns of different indexes, include [22], [23] and [24].

3.3 ASYMMETRIC-GARCH MODEL

[25] demonstrates that a period with a negative movement of the market price has a higher influence on the value of volatility in the next period than equivalent positive change. That difference in weight assigned to different residuals, based on their sign, is also known as asymmetry. [26] accordingly responded to the realized asymmetry by modifying the GARCH formula as:

$$h_t = \omega + \alpha e_{t-1}^2 + \alpha^{-} S_{t-1} e_{t-1}^2 + \beta h_{t-1} \tag{26}$$

Where

$$S_t = \begin{cases} 1 \text{ if } e_{t-1} < 0 \\ 0 \text{ if } e_{t-1} > 0 \end{cases} \tag{27}$$

In the case of price fall, it provides an additional weight of α^{-} to the residual e_{t-1}^2, to cover that difference it has on the change in the coming period. This method is known as GJR-GARCH or as GJR (1,1). In other words, the squared residual is given the weight of α in the period following the return above its conditional expectation, whereas the weight of $\alpha + \alpha^{-}$ otherwise.

General GJR-GARCH(p,q), would be used to forecast the next period by:

$$h_{t+1} = \omega + (\alpha + \alpha^{-} S_t) z_t^2 + \beta h_t \tag{28}$$

$$E[h_{t+1} | I_{t-1}] = \omega + (\alpha + \alpha^{-}/2 + \beta) h_t \tag{29}$$

E $[S_t]$=0.5 is assumed through considering an equal likelihood of preceding residual having positive or negative value and accordingly for the value of S_t to be 0 or 1.

While [27] argue that some other models, with a focus on QGARCH, outperform GJR, [28] and [29] claim that GJR-GARCH outperforms other methods when applied on stock indices.

3.3 GJR WITH CONDITIONAL T-DISTRIBUTION

As many researchers confirm that returns are Non-Gaussian, considered were various alternative distributions. In that direction, empirical evidence was provided by [30] and [31], contradicting the assumption that returns follow a conditional normal distribution. Among the alternatives for N(0,1) [31] suggested the standardized t-distribution.

As such, in [31] the calculation was done by using the GARCH(1,1) model. The conditional distribution of the model is considered to be of standardized t-distribution, with ν degrees of freedom. The proper value of ν would also be determined through the optimization process, as to determine the optimal one. The variable ν is not present either in the constraints of the process, nor in any of the calculations of other variables, but in the objective function which is due to be maximized. The objective function used for t-distribution, used for the same purpose by [32], is:

$$l_t = -\frac{1}{2}\ln(h_t) + \ln\big(c(v)\big) - \frac{v+1}{2}\ln\left(1 + \frac{z_t^2}{v-2}\right) \quad (30)$$

Where $c(v)$, based on the gamma function, is defined as:

$$c(v) = \frac{\Gamma\left(\frac{1}{2}(v+1)\right)}{\Gamma\left(\frac{1}{2}v\right)\sqrt{\pi(v-2)}}. \quad (31)$$

With gamma function being of integral type:

$$\Gamma(y) = \int_0^\infty x^{u-1}e^{-x}dx, \quad u > 0. \quad (32)$$

Those include four American Indexes of Standard & Poors 500, NASDAQ, Dow Jones Industrial, and New York Stock Exchange Composite Index (NYA). Added to that list are four European Indexes of British “The Financial Times Stock Exchange” FTSE100, French “Cotation Assistée en Continu” CAC40, German DAX, and Spanish IBEX35. Moreover, analyzed are three Asian Indexes of Japanese Nikkei, Chinese HANG SENG, and Indian BSE30.

4. TEST ON S&P INDEXES

In this part the introduced models of GARCH(1,1), and two models of GJR(1,1) will be tested on a real market data of daily closing price of Standard and Poor’s Index, and their outcomes compared.

4.1 GARCH (1,1) TEST ON S&P 100

To apply the GARCH(1,1) used is Standard and Poor’s 100 Index p(t) set of historical daily closing prices, from January-2006 until June-2016. The daily return rate is calculated using a continuous percentage change formula (1). In the equation of conditional variance h_t, the first day gets approximated by calculating sample standard deviation to the power of two. For the trading days that follow, it gets calculated by using the equation (13). Within the process, variables of $\omega, \alpha,$ and β get assigned some initial values, which (in addition to μ) get modified by reassigning them the proper values through the process of optimization. Calculation of z_t is done by (9) assigning an initial value that would be changed by the optimization process. Finally, the process gets optimized through maximizing the objective function of

$$\text{Log L} = \sum_{t=1}^{n} -\frac{1}{2}[\log(2\pi) + \log(h_t) + z_t^2]. \quad (33)$$

The constraints included in the optimization process are those having $\alpha + \beta < 1$, and having none of the variables ω, μ, α and β negative. The optimization outcome appears in Table 1.

Table shows on the left-hand side some essential statistical characteristics of the analyzed daily returns, while the right-hand side shows the outcomes achieved by the already described method of GARCH (1,1). The long-term volatility measure gets calculated by assigning the optimization process outcomes of ω, μ, α and β in the formula (23).

The calculation provided an outcome of one-day measure for long-term volatility, the value of 0.01131 or 1.131%. The corresponding annual volatility gets attained by multiplying the daily volatility value by the square root of 252 (average number of trading days per year in the US financial markets, after exclusion of holidays and weekends) reaching the value of 0.179525 or 17.95%. As for any day t, the conditional variance gets estimated by using the outcomes of the preceding trading period and applying it in the GARCH formula through the optimization outcomes as:

$$h_t = 2.3273 \times 10^{-6} + 0.11554(r_{t-1} - 0.0006532)^2 + 0.866263h_{t-1}. \quad (34)$$

Table 1 GARCH (1,1) optimization outcomes for the S&P100

CHARACTERISTICS OF DAILY RETURNS r(t)	
Mean	0.00025991
St.Dev.	0.01254081
Skewness	-0.0361637
Kurtosis	10.8230376
Autocorrelation	-0.1113227
Minimum	-0.087769
Maximum	0.1124342
GARCH (1,1) OUTCOMES	
M	0.0006532
ω	2.3273E-06
α	0.11554
β	0.866263
Persistence	0.981813
LONG TERM VOLATILITY	
Daily	Annualized
0.01131	0.179525

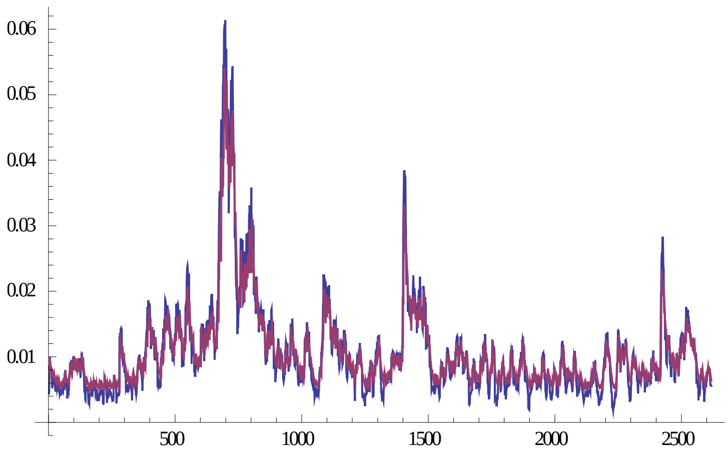


Figure 1: Daily volatility by GARCH (1,1) and historical average

The most straightforward comparison of the outcomes would be with the primary historical volatility. It is calculated by taking a few returns of preceding periods and calculating their standard deviation. Figure 1 shows the two outcomes of the variance, the first estimated by the GARCH(1,1), and the second by the historical variance using the twenty-five days history of squared residuals. The red graph displays the GARCH(1,1) estimation, while the blue one represents the historical average.

Figure 1 displays that the historical-average daily measure of volatility that generally exceeds the equivalent GARCH estimation, on both upper and lower sides. That obvious conclusion gets confirmed by the difference of standard deviation values of the two estimations. As the calculated daily mean-historical-volatility has the standard deviation of .00037, the corresponding GARCH has it of .000277.

Figure 2 displays the annualized GARCH(1,1) calculated daily conditional volatility, and compared to the corresponding volatility of “VXO” provided by The Chicago Board Options Exchange for S&P 100.

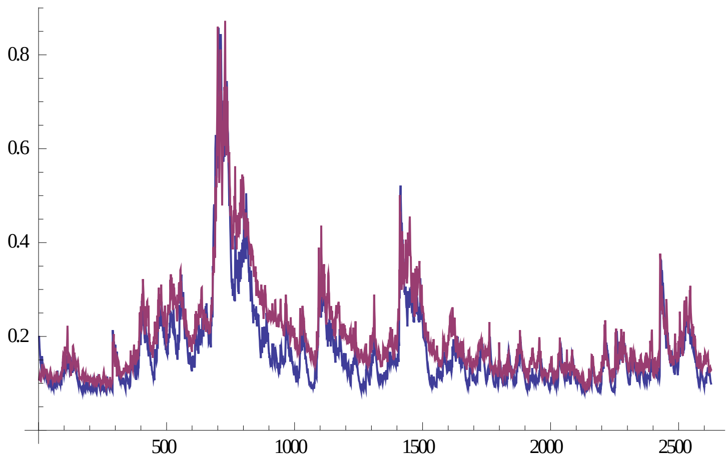


Figure 2: The GARCH estimated Volatility and VXO

In the implied volatility calculation process of both VIX and VXO, CBOE takes into consideration is the time to maturity of one month and uses the at-the-money put and call options. Since the GARCH (1, 1) calculation is for S&P100, the corresponding volatility estimation provided by the market is VXO. Through simple visual analysis of Figure 2, it shows some similarity in the estimated deviation measure, while in general VXO is higher than the GARCH forecast.

Observed in Figure 3 are the outcomes of the standardized residual, calculated by dividing the daily residual of return by the estimated volatility. In a Gaussian distribution, the value of standardized residual is expected to move between the values of -3 and 3. It is evident that the daily standardized residual for S&P100 exceeds that range more frequently. As in a normal distribution, it would occur less than three times in a thousand trials for z-value to exceed the range of (-3, 3) while once in two thousand trials to exceed the range of (-3.5, 3.5).

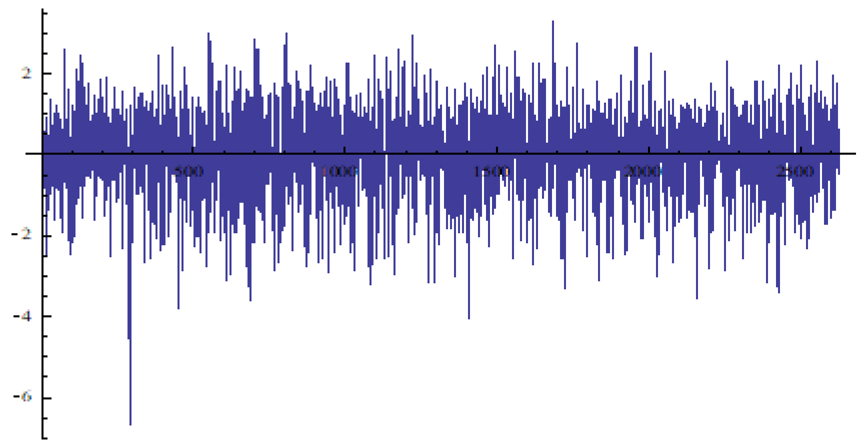


Figure 3: The daily standardized residual estimated by GARCH(1,1)

In the 2625 trading days of S&P100, the standardized residual, which gets calculated within the GARCH (1,1), occurs to be outside the normal range of (-3, 3) nineteen times. Twice the z value turns out to be higher than 3, while the remaining 17 times its value is less than -3. Also, the negative residuals have more extreme values, as of the seventeen values of the standardized residual being less than negative three, six are further below -3.5. A similar outcome of having the standardized residual results higher than 3.5 occur zero times for the positive residuals.

4.2 TEST OF ASYMMETRIC MODEL ON S&P 100

The GJR(1,1)-MA(1)-M model combines the three characteristics, including GJR(1,1), and M for the inclusion of conditional variance of h_t within the formula of mean-return μ_t. The third charachteristic is that of MA (1), represents a moving average that is also a part of ARMA and GARCH but in this case included in the formula (37) of daily mean-return as Θe_{t-1}. (Taylor 2011)

$$h_t = \omega + (\alpha + \alpha^{-} S_{t-1}) e_{t-1}^2 + \beta h_{t-1} \tag{35}$$
$$r_t = \mu_t + e_t = \mu_t + z_t \sqrt{h_t} \tag{36}$$
$$\mu_t = \mu + W\sqrt{h_t} + \Theta e_{t-1} \tag{37}$$

Accordingly, the GJR (1,1) part of the calculation process is present in the formula for the conditional variance h_t (35). Similarly, in the equation (37), the presence of h_t adds to it the M part. Finally, presented is the MA (1) part of Θe_{t-1} in the equation (37) for evaluation of the variable of μ_t.

Applying the method through the tools of Microsoft Excel, using the daily closing price of Standard & Poor's 100 Index for more than 2600 trading days, shown are the attained optimization results in Table 2.

Table 2 Outcome of GJR (1,1)-MA(1)-M model with conditional normal distributions

GJR(1,1)-MA(1)-M OUTCOMES	
μ	1.0018E-07
W	0.05027265
Θ	-0.02126269
Ω	1.4995E-06
α	0.04032983
α-	0.05846881
β	0.91356802
Persistence	0.983132
Long-Term Volatility	
Daily	Annual
0.0094286	0.149675

After placing the outcomes from table 2 instead of their variables of equations (35) and (37), provides the corresponding equations (38) and (39). These equations make it simple to estimate both the conditional variance and the expected return for the next period.

$$h_t = 1.4995 \times 10^{-6} + (0.04033 + 0.05847 S_{t-1}) e_{t-1}^2 + 0.9136 h_{t-1} \tag{38}$$
$$\mu_t = 1.0018 \times 10^{-7} + 0.05027\sqrt{h_t} - 0.02126 e_{t-1} \tag{39}$$

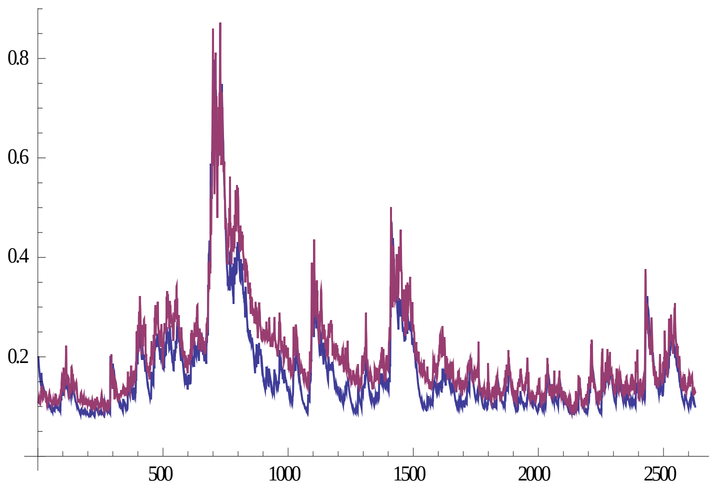


Figure 4: GJR (1,1)-MA(1)-M Volatility estimation and VXO

Figure 4 shows the GJR(1,1)-MA(1)-M daily conditional volatility estimation compared to the that of VXO, where just as in the previous comparison with the GARCH(1,1) estimation, GJR model generally underestimates the daily volatility.

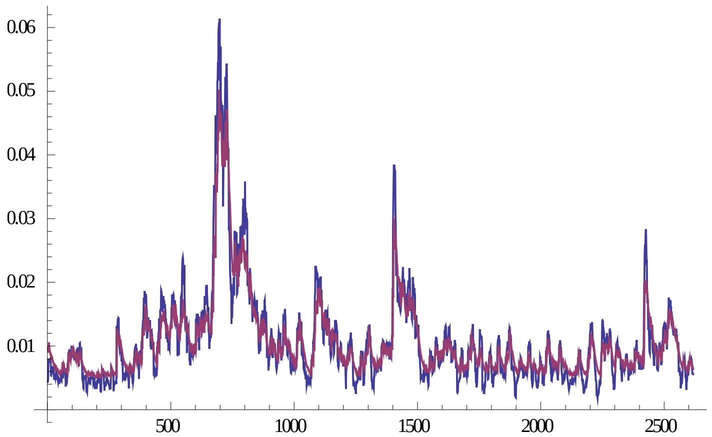
Table 2 GJR (1,1)-MA(1)-M model optimization outcomes

GJR(1,1)-MA(1)-M	**Normal Dist.**		**T Distribution**	
μ	1.0018E-07		0.000145977	
W	0.05027265		0.061944561	
Θ	-0.02126269		-0.06699483	
ω	1.50E-06		2.35785E-06	
α	0.04032983		0.01743715	
α-	0.05846881		0.199096732	
β	0.91356802		0.857732023	
ν			6.257815	
Persistence	0.983132		0.974718	
Long-Term Volatility	Daily	Annual	Daily	Annual
	0.009429	0.1497	0.009657	0.1533

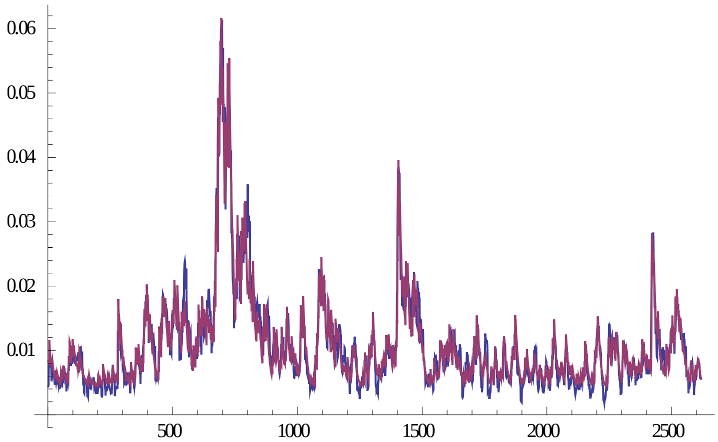
While showing similarity in the estimation of the values of long-term volatilities, both daily and annual, under different distributions considered, all the other variables provided by the optimization process have reasonably different values. Further analyses, which follow, describe the actual differences between them.

Figure 5 shows the visual form of the GJR (1,1)-MA(1)-M models volatility estimation, with different returns' distribution assumptions, compared to the historical volatility means. Considered under normal conditional distribution, the outcomes of GJR(1,1)-MA(1)-M show outcomes very similar to those of GARCH(1,1), with values graphically seen to fall within the limits of historical mean. The calculation of GJR(1,1)-MA(1)-M with the T-distribution differs more from GARCH(1,1).

Confirmation of the same is possible with the standard deviation values of the daily estimated variances. As the standard deviation value of the GJR(1,1)-MA(1)-M with the normal conditional distribution is 0.000267, the GJR(1,1)-MA(1)-M with t-distribution has 0.000331. Results, of non-normal distribution, being closer to the standard deviation value of 0.000371 calculated by the ten days historical mean volatility estimation.



(a)



(b)

Figure 5: The Difference between the ten-days mean historical volatility and GJR(1,1)-MA(1)-M estimation with the conditional distribution assumed to be (a) Gaussian, (b) T-distribution.

As outcomes in table 4 show, α and α- are among the variables with a significant change in value between the estimates under the two different distribution assumptions for returns. The change in distribution assumption corresponds significantly to the asymmetry ratio A, calculated by A= (α + α-)/ α. While, under the normal distribution, the optimization outcomes give the asymmetry ratio of 1.49. The equivalent asymmetry ratio has values higher than 10 under the t-distribution.

The daily standardized residual gets calculated by equation (9). As far as the fat tail feature of volatility is concerned, the proportion of standardized daily residuals exceeding the range of (-3, 3) did not change significantly between the two models. The small change that took place was in the unexpected direction, where the number of occurrences outside the range of (-3, 3) increased from 20 under a normal distribution to 22 under T-distribution.

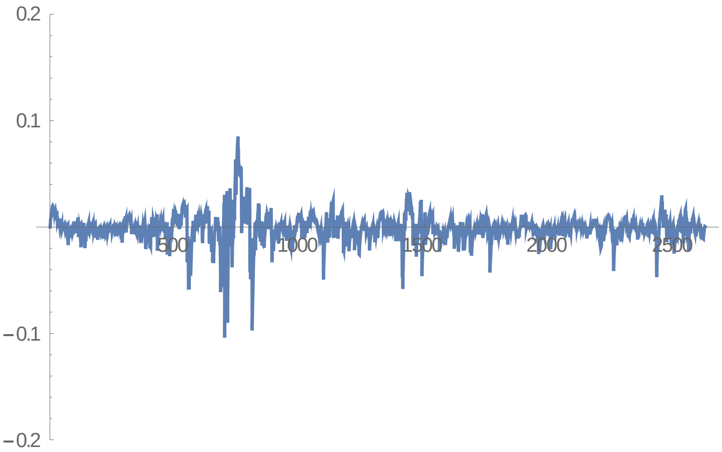
Table 3: Frequencies of S&P100 standardized daily returns estimated by GJR(1,1)-MA(1)-M

	GJR with Normal Distribution	GJR T-Distribution	
Range	observed-normal	observed-normal	Normal
0 to 0.25	3.19	2.92	9.87
-0.25 to 0	2.28	2.62	9.87
0.25 to 0.50	2.03	1.23	9.28
-0.50 to -.25	-0.14	-0.06	9.28
0.50 to 1.00	-0.75	-0.29	14.99
-1.0 to -.50	-2.54	-1.81	14.99
1.0 to 1.50	-0.39	-1.49	9.18
-1.50 to -1.00	-2.94	-2.44	9.18
1.50 to 2.00	-0.94	-1.51	4.41
-2.00 to -1.50	-1.09	-0.40	4.41
2.00 to 3.00	-0.35	-0.54	2.14
-3.00 to -2.00	1.14	1.21	2.14
> 3	-0.021	0.017	0.13
< -3	0.51	0.55	0.13
Sum(ABS)	18.83	17.12	

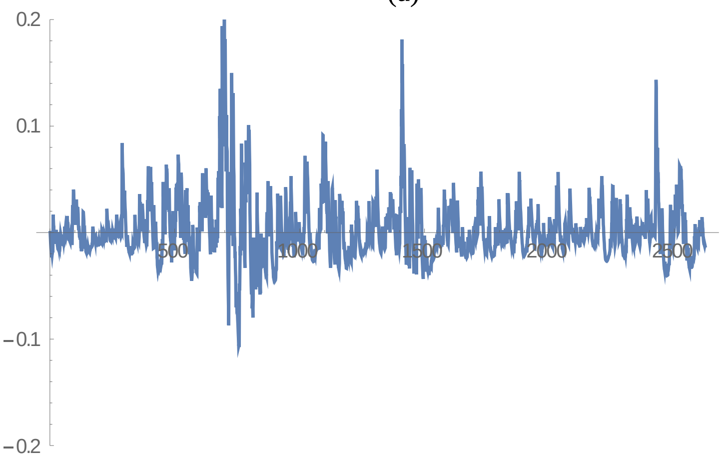
The mean value is among the core characteristics of these outcomes of the daily standardized residual, z_t, for the two different calculations of GJR(1,1)−MA(1)−M models. As in the standard normal distribution, it is supposed to be zero; it turns out to be -0.028, -0.060 respectively for normal and T conditional distributions. As these values do not seem to be at a significant distance from the normal distribution, a similar conclusion gets confirmed through the corresponding standard deviation values of 0.993469 and 0.99562, which in normal distribution would be 1.

As can be expected from the mean values, both are skewed to the left, having very close values of -0.499084 and -0.49875. As all of these details confirm the distributions distance from the normal distribution, the values of their kurtosis are significantly different from the Gaussian distribution value of 3, instead 4.5403 and 4.4604, respectively.

As the kurtosis value shows a slight shift towards the normal distribution, as being closer to 3, reached is a similar conclusion through revising the Table 4. This table shows the occurrence frequencies of the standardized residual for 14 different intervals for each of the three tested distributions. Calculated is the percentage of occurrence of standardized residual values of the estimated 2625 trading days and their belonging within those intervals, and compared to the corresponding proportion under the normal distribution. The outcome shows varying differences that can be identified for those ranges, looking at each 14 of them. In total, the sum of absolute values of differences of 18.833 and 17.116 for each.



(a)



(b)

Figure 6: Differences between the annualized estimated daily conditional volatility (a) $\sqrt{h}_{GJR(1,1)MA(1)M-T} - \sqrt{h}_{GARCH(1,1)-NORMAL}$; (b) $\sqrt{h}_{GJR(1,1)MA(1)M-T} - \sqrt{h}_{GJR(1,1)MA(1)M-NORMAL}$.

Figure 6 included the difference between the annualized daily conditional volatilities estimated by different ARCH methods. Part (a) of the figure shows the difference between the annualized conditional volatility daily estimations of GARCH(1,1) and GJR(1,1)-MA(1)-M, both under conditional normal distribution. Looking at (b), the difference between the volatility estimates of GJR(1,1)–MA(1)-M under normal distribution versus the corresponding GJR(1,1)–MA(1)-M under T-distribution is significantly greater than its difference versus GARCH(1,1).

Is any of them correct? A right choice needs to be selected. Given the availability of intraday data for S&P500 Index, the same calculations were repeated using the daily closing price of S&P500, with a slight change in sample size. To avoid extreme values mostly experienced through the period of global financial crises, the sample starts from mid-2009 and includes 1825 daily closing prices. As for the stated S&P500 data, the optimization outcome of GARCH (1,1) was:

$$h_t = 4.0148 \times 10^{-6} + 0.1412(r_{t-1} - 0.0002316)^2 + 0.81947h_{t-1}. \tag{40}$$

The same data used for GJR(1,1)-MA(1)-M model, with conditional normal distribution, provided the outcomes of:

$$h_t = 3.91879 \times 10^{-6} + (0.01411 + 0.20879S_{t-1})e_{t-1}^2 + 0.8338h_{t-1}, \tag{41}$$

with the corresponding mean return:

$$\mu_t = -0.00018514 + 0.08365\sqrt{h_t} - 0.0378e_{t-1}. \tag{42}$$

Similar optimization for GJR(1,1)-MA(1)-M model, with conditional T-distribution, provided the estimates of:

$$h_t = 3.45467 \times 10^{-6} + (0.020274 + 0.24188S_{t-1})e_{t-1}^2 + 0.8227h_{t-1}, \tag{43}$$

in addition to its periodic value of the mean return:

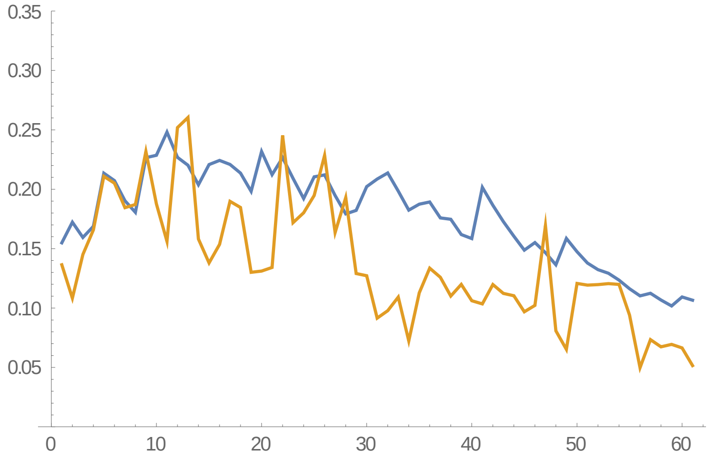
$$\mu_t = -2.2873 \times 10^{-5} + 0.09945\sqrt{h_t} - 0.0403e_{t-1} \tag{44}$$

Those volatility estimation formulas were used to calculate the daily conditional volatility all through the studies period. Throughout the in-sample testing period of the 61 trading days at the beginning of 2016 for which the daily volatility was calculated using the intraday price of the S&P500 Index. The data included prices on the 15-minutes interval and the volume of trade for each of them.

Consequently, each of these intervals is assigned the weight of its trading volume as compared to the total daily trade of that asset. Accordingly, calculated are the daily weighted mean and standard deviation. The calculated values were further modified by multiplying by the square root of the number of trading days in a year, to have the annualized volatility measure.

Part (a) of Figure 10 shows the outcomes of the GJR(1,1)-MA(1)-M models and the realized volatility which is visually detachable from the other two. As in the most of the 61 trading days, the realized volatility shows to be below all three of the estimates. Also, evaluated by eyesight, the normal distribution seems to outperform the T-distribution.

(a)



(b)

Figure 7: Visually displaying the outcomes of realized volatility as compared to (a) GJR(1,1)-MA(1)-M models with different distribution assumptions; (b) GARCH(1,1).

Part (b) of Figure 7 shows the difference of the GARCH (1, 1) estimation and the daily realized volatility. Through a visual comparison, the GARCH (1, 1) estimations are closer to the calculated realized volatility than either of the GJR models shown in part (a). Despite its inclusion of additional parameters, the asymmetric model of GJR fails to outperform the simpler GARCH, suggesting that the asymmetry measure identified through ten years used in the optimization process did not hold in the testing period.

5 CONCLUDING REMARKS

The parametric methods of ARCH went through lengthy research, and developed are tens and tens of different models. From the aspect of practicality, applying either of them is very demanding, they try to implement as many of the learned characteristics of the market price in the process. While considering its coverage of the confirmed asymmetry characteristics, GJR-GARCH would be a preferable choice among the revised ARCH models.

Included in this study is are the two models of GJR(1,1)-MA-M differing in the assumption of its conditional distribution that, as opposed to the GARCH(1,1) model applied, do not have a constant daily mean value of return. Instead, as explained in the text, the value of μ is calculated for each period separately as μ_t. Despite such flexibility, estimates by GARCH(1,1) with conditional normal distribution are, for the first 30 trading days, closer to the realized volatility than either the GJR-MA-M with conditional normal distribution or the GJR-MA-M with conditional T-distribution.

Used for test and comparison is a sample of three months of daily intraday prices of Standard and Poor's 500 Index, and their corresponding daily standard deviation. For each of these days calculated is the corresponding conditional variance, using the developed formulas of 40-44. The calculated outcomes, as shown in Figure 7, suggest that a simple GARCH(1,1) can outperform a further developed models of GJR(1,1). That could suggest a significant change in the asymmetry between first thirty days of the test period and the interval used in the optimization process.

At the same time, the sample shows that GJR-MA-M model does not show a significant difference, comparing the two estimates under different assumptions of conditional distribution.

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