

Applications of Mathematics: Dispersion of soluble matter in solvent flowing through a tube under a steady pressure gradient

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ABSTRACT

This paper contains many directions such as Estuary, Solute, Concentration, and Diffusion. Including a sample tube under a steady pressure gradient, the partial differential equations, and the ordinary differential is following the solution. The result comes from the reality that a probability of a mote crossing out of the element of the cross-sectional area of the flow is independent of some position of the cross-sectional element. Thus, the solute spread over a distance proportional to $t^{1/2}$.

Keywords: Solute, Estuary, Concentrations, Diffusion, and partial differential equations

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1. Introduction

The unstable flow through a pipe of uniformly cross-sectional under the influence of a statistically constant pressure gradient is both statistically constant and function of the radial coordinate only[1]. In constant flow laminar or statistically constant unstable flow through a pipe of uniformly cross-sectional, an injected material spreads along a pipe in the direction of the flow due to longitudinal effects of molecular or turbulent diffusion and the interaction between the convection and the lateral molecular or turbulent diffusion. This happens because, at the wall, the velocity falls to zero and increases gradually reaching maximal at the tube center (or at the free Surface in an estuary), Thus a part of the injected material which is initially near the tube center (or at the free surface in an estuary) will be carried along the tube (or estuary) faster than parts which are close to the walls[2-4]. But eventually, the concentration becomes uniform over the Cross-Section due to lateral molecular or turbulent diffusion. When Taylor[5, 6] gave the basic idea of this mechanism, he assumed that the injected material and solvent are of the same density. Thus, the particles of the injected material are hydrodynamically indistinguishable from the solvent particles and, from the definition of the discharge velocity of the Whole fluid in the tube, the velocity means of the injected particles is equal to the discharge velocity. Rather, the result is the result from the reality that the probability of a particles slice through a given cross-sectional area element of the flow, is independent of the position of cross-sectional element,[7-13]. Moreover, the velocity mean of the center from a cloud of the injected substance is equal to the velocity of discharge[14-16]. This analysis will be useful when we use axes moving with discharge velocity.

1.1 Presentation of the problem

This problem relates to a mathematical model of the interaction between fluid flows estuary, solvent, solute, concentration, and diffusion.

1.2 Objective of the project

The main objective of this paper is:

- 1) Define Estuary.
- 2) Apply the Estuary and its contents.
- 3) This analysis will be useful when we use axes moving with the discharge velocity.

2. Dispersion in a laminar flow through a tube

Consider a straight circular tube of radius a in a fully developed laminar flow when the concentration is axisymmetric. The longitudinal velocity u at a distance r from the centerline of the tube is:

$$u = \bar{u}(1 - 2\frac{r^2}{a^2}) \quad \dots(1)$$

Measured relative to axes moving with the discharge velocity of the flow \bar{u} . The equation that governs the distribution of the concentration $c(x,r,t)$ can be written in the form:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x_1} = \frac{k}{a^2} \left(\frac{\partial^2 c}{\partial z^2} + \frac{1}{z} \frac{\partial c}{\partial z} + a^2 \frac{\partial^2 c}{\partial x_1^2} \right) \quad \dots(2)$$

Where $z = \frac{r}{a}$, $x_1 = x - \bar{u}t$ and the coefficient of molecular diffusion are assumed to be independent of the concentration. In using equation (2) we assumed there are no sources of matter in the tube.

The boundary condition of no flux at the wall is:

$$\frac{\partial c}{\partial z} = 0 \text{ at } z=1 \quad \dots(3)$$

Hence the mean concentration is easier to measure than the concentration. The mean concentration $\bar{c}(x_1,t)$ is given by:

$$\bar{c} = \frac{2}{a^2} \int_0^a cr \, dr \quad \dots(4)$$

The cross-sectional average of equation (2) is:

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial}{\partial x_1} (\bar{u}\bar{c}) = k \frac{\partial^2 \bar{c}}{\partial x_1^2} \quad \dots(5)$$

the goal of our analysis now is to reach an expression for the rate of transport $\bar{u}\bar{c}$, which we can substitute into equation (5). There is an exact steady solution of equation (2) in which the concentration along the tube axis is exactly linear in x_1 . i.e.

$$c = \alpha x_1 + \alpha g(z) \quad \dots(6)$$

Where α is a constant. In the real situation, over time after lunch, the tint cloud will spread over a distance that is greater than the radius. Thus, at about any point, the concentration will be approximately linear in x_1 . So, we can use equation (6) to find an approximation to $\bar{u}\bar{c}$ in a real situation.

Now substitute equation (6) into equation (2) yielding, after one integration

$$z \frac{dg}{dz} = \frac{a^2 \bar{u}}{k} \left(\frac{z^2}{2} - \frac{z^4}{2} \right) + A,$$

Where A is a constant. From the boundary condition (3) A is equal to zero. If we integrate again,

$$g = \frac{a^2 \bar{u}}{4k} \left(z^2 - \frac{z^4}{2} \right) + B \quad \dots(7)$$

Where B is a constant. From the exact solution (6) it follows that $\alpha = \frac{\partial \bar{c}}{\partial x_1}$, so that from (6) and (7)

$$\alpha x_1 = \bar{c} - \frac{\partial \bar{c}}{\partial x_1} \left(\frac{a^2 \bar{u}}{12k} + B \right) \quad \dots(8)$$

There for

$$c = \bar{c} - \frac{\partial \bar{c}}{\partial x_1} \left\{ \frac{1}{3} - z^2 + \frac{1}{2} z^4 \right\} \frac{a^2 \bar{u}}{4k} \quad \dots(9)$$

[Note that equation (9) is independent of the value of the constant B in equation (7)]

The mean rate of transport $\overline{u\bar{c}}$ across a section is:

$$\overline{u\bar{c}} = 2 \int_0^1 cuz \, dz$$

Thus, from equation (1) and (9)

$$\overline{u\bar{c}} = - \frac{a^2 \bar{u}^2}{48k} \frac{\partial \bar{c}}{\partial x_1} \quad \dots(10)$$

That is equation (5) becomes.

$$\frac{\partial \bar{c}}{\partial t} = \left(k + \frac{a^2 \bar{u}^2}{48k}\right) \frac{\partial^2 \bar{c}}{\partial x_1^2} = K \frac{\partial^2 \bar{c}}{\partial x_1^2} \quad \dots(11)$$

Thus, the mean concentration $\bar{c}(x_1, t)$ is scattered comparative to a plane which moves with the discharge velocity \bar{u} and satisfies a diffusion equation with an effective longitudinal diffusion parameter k, where k is given by:

$$K = k + \frac{a^2 \bar{u}^2}{48k} \quad \dots(12)$$

The value of K in equation (12) is due to Aris [17].

Taylor [1] and [18-20] ignored the effect of the longitudinal diffusion in comparison with the convective diffusion and found $K = \frac{\bar{u}^2 a^2}{48k}$; however, since $\frac{\bar{u}a}{k} \gg 1$ in most practical cases, Taylor's result is an acceptable approximation.

Equation (11) holds with increasing accuracy a percentage of time after injection to the cross-sectional mixing time increases as $\frac{kt}{a^2}$ increases [21].

The appropriate solution of (11) for a cloud of dye is the normal curve.

$$\bar{c}(x_1, t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(x-\bar{x})^2}{2\sigma^2} \right] \quad \dots(13)$$

Thus, the dye cloud eventually spreads symmetrically around a point moving with the flow discharge velocity \bar{u} , for this reason, its center of mass \bar{x} and its standard deviation σ satisfies:

$$\begin{aligned} \frac{d\bar{x}}{dt} &= \bar{u} \\ \frac{d\sigma^2}{dt} &= 2k \end{aligned} \quad \dots(14)$$

Experimentally Taylor [1] and analytically, [14] showed that if sufficient time has not passed, the symmetry of the distribution of concentration is not completely smoothed out. The standard deviation σ is a measure of linear extension, which implies that $\sigma = \text{spread}$, where $\sigma = \sqrt{2kt}$ by equation (14) thus the solute is spread over a distance proportional to \sqrt{t} .

3. Dispersion in a turbulent flow through a tube

In a consideration of turbulent flow, it is necessary to refer to the fact that the velocity and the pressure gradient are not constant in time as in the laminar flow, but they are a random function of it. Thus, the statistical mean velocity is constant only when the flow is statistically stationary. The distribution of the longitudinal velocity is considerably more uniform over the cross-section due to the increased transfer of momentum in the transverse direction.

The purpose of this section is to see whether the mechanism described earlier lead to an equation of the form (11) when the turbulence velocity is a stationary random function of time.

Consider a statistically the turbulent flow is steady through a pipe of uniform cross-section with coordinates. Experimentally, the statistical mean velocity u at a distance r from the center of the tube [22, 23] satisfies:

$$u = u_0 - u_* f(z) \quad \dots(15)$$

where $f(z)$ is the same for all straight tubes with a circular cross-section, at the center of the tube, u_0 is the velocity and $u_* = \left(\frac{T_0}{\rho}\right)^{\frac{1}{2}}$, T_0 and ρ being the frictional stress at the wall of the tube, and the fluid density, respectively. The discharge velocity is:

$$\bar{u} = 2 \int_0^1 uz \, dz,$$

and by using equation (15)

$$\bar{u} = u_0 - 2u_* \int_0^1 z f(z) \, dz .$$

The integral $\int_0^1 z f(z) \, dz$ was evaluated numerically by Taylor [1] using the value of $f(z)$ determined experimentally by and by [24-26] giving:

$$\bar{u} = u_0 - 4.25u_* \tag{16}$$

The mass balance equation that governs the distribution of the concentration $c(x,r,t)$, with axes moving with the discharge velocity, can be written in the form:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x_1} = \frac{1}{r} \frac{\partial}{\partial r} (e_1 r \frac{\partial c}{\partial r}) + \frac{\partial}{\partial x_1} (e_2 \frac{\partial c}{\partial x_2}) \tag{17}$$

Where e_1 and e_2 are the eddy diffusivities in the r and x_1 directions, respectively.

The contribution from the longitudinal turbulent diffusion (x_1 - direction) is not large and can be ignored temporarily, i.e., we ignore the term $\frac{\partial}{\partial x_1} (e_2 \frac{\partial c}{\partial x_2})$.

Thus equation (17) becomes:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x_1} = \frac{1}{r} \frac{\partial}{\partial r} (e_1 r \frac{\partial c}{\partial r}) \tag{18}$$

The cross-sections average of equation (18) is:

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial}{\partial x_1} (\bar{u}\bar{c}) = 0 \tag{19}$$

now it is necessary to find the approximate expression for $\bar{u}\bar{c}$ as in the case of laminar flow. We use an expression for the eddy diffusivity e_1 based on Reynold analogy.

Thus e_1 can be expressed in the present case by the equation.

$$e_1 = - \frac{T}{\rho \frac{\partial u}{\partial x}} = - \frac{M}{\frac{\partial c}{\partial x}} \tag{20}$$

Here T is the turbulent shear stress (arising from fluctuations in the velocity carrying fluctuations in longitudinal momentum radially) and M is the turbulent rate of transfer of matter radially (arising from fluctuations in velocity carrying fluctuations in concentration radially). Both T and M can be expressed in terms of the fluctuations using the classical Reynolds decomposition. See for the instance the survey by Tennekes and lunley [27].

Hence using (15) the coefficient e_1 can be written as:

$$e_1 = \frac{\partial u_* z}{f'(z)} \tag{21}$$

As in the laminar case, equation (18) has an exact steady solution of the form (6). Using equation (16), equation (18) becomes:

$$\frac{d}{dz} \left(\frac{z^2}{f'(z)} \frac{dg}{dz} \right) = za \{f(z) - 4.25\}$$

By integration we obtain

$$g = a \int_0^z \frac{f'(y)}{y^2} \left[\int_0^y x(f(x) - 4.25) \, dx \right] dy \tag{22}$$

proceeding as in the laminar case, we find that.

$$\overline{uc} = -10.06 au_* \frac{\partial \bar{c}}{\partial x_1} \quad \dots(23)$$

Where integrals like $\int z(f(z) - 4.25)dz$ etc, have been determined by numerical integration. Therefore equation (19) becomes:

$$\frac{\partial \bar{c}}{\partial t} = 10.06 au_* \frac{\partial^2 c}{\partial x_1^2} = k_1 \frac{\partial^2 \bar{c}}{\partial x_1^2} \quad \dots(24)$$

Thus, the mean concentration $\bar{c}(x_1, t)$ satisfies the diffusion equation with dispersion coefficient k_1 , where k_1 is given by.

$$k_1 = 10.06 au_* \quad \dots(25)$$

And the injected material is scattered comparative to a plane which moves with the discharge velocity \bar{u} . As in the laminar cases, equation (24) adheres with an accuracy increasing as a proportion of time after injection to the cross-sectional mixing time increases i.e., as $\frac{u_* t}{a}$ increases.

We have ignored the longitudinal turbulent diffusion in our treatment a reasonable assumption to find this contribution is that the coefficient of longitudinal turbulent diffusion, i.e., $e_1 = e_2$ which is valid if the turbulence is isotropic. The rate of transfer of matter across a plane due to the longitudinal turbulence is:

$$\underline{Q} = \int_0^a 2\pi e_1 \frac{dc}{dx} = 2a^3 u_* \frac{dc}{dx} \int_0^1 \frac{z^2}{f'(z)} dz$$

The value of the integral can be found numerically. The mean coefficient of the diffusion due to the longitudinal turbulent (x-direction) velocity satisfies.

$$\underline{Q} = k_2 \pi a^2 \frac{dc}{dx}$$

where k_2 is the longitudinal turbulent diffusion. Thus

$$k_2 = 0.052 au_* \quad \dots (26)$$

Therefore, the total longitudinal turbulent diffusion due to both effects is from (25) and (26)

$$K = 10.1 au_* \quad \dots(27)$$

Equation (24) has of course a Gaussian solution as in the laminar case. Experimentally, Taylor [2] found values of k from 11.0 to 12.8 for the straight tube. These values seem to be in good agreement with the theoretical value given by (27). However other work in this field [14] casts doubt on the accuracy of the eddy diffusivity approximation and Reynold's analogy and points out the consequences of ignoring the viscous sub-layer. The taken time by dye of molecule to sample all the cross-sectional including the viscous sub-layer is greater than $\frac{a}{u_*}$. This occurs because the viscous sublayer properties of the disorder directly dependent on viscosity ν and lateral mixing is sufficiently dominated near the wall by molecular processes, the intensity of which is measured by molecular diffusion k .

Thus, the longitudinal dispersion coefficient is given in equation (27) must increase by a moment proportional to the size of the viscous sub-layer. It has been suggested that error arising is of order 10-20% (depending on the Reynolds number) which seems to be an important effect; even this estimation is based on Reynold's analogy which may also cause an error of order 10%. Moreover, the values of k in rivers and canals, whenever the width-to-depth ratio is large, are far larger than (27) because in such cases the dominant dispersion mechanism is due to the transverse variation of velocity.

4. Conclusions and recommendations

This paper is important to research for the Universities, especially in the engineering college's student's projects. The applications in the rivers and seas such as estuary, solute, concentration, and diffusion with the use of ships and steamship and submarine, are fields of the applications with the salinity of this research and all under partial and ordinary equations.

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