Subsequences of real valued sequences and convergence

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ABSTRACT

Convergence of real valued sequences is a classical subject of study for many mathematicians and it continues to be studied in recent years. Different types of convergence including ordinary, statistical, almost and ideal convergence, and related properties have been researched. In some studies, the relationship of a sequence and its subsequences regarding some type of convergence is investigated, using Lebesgue measure and Baire category. In this paper we revisit ordinary convergence and study the properties of subsequences of a real valued sequence using measure and category. We state and prove some simple theorems offering fresh insights into a classical subject.

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1. Introduction

Convergence of real valued sequences has been a subject of study for many mathematicians over the last century and in recent years. Many different types of convergence including ordinary, statistical, almost an ideal convergence, and related properties have been researched. In some studies, the relationship of a sequence and its subsequences regarding some type of convergence was investigated. For this purpose, two different gauges of size were used: Lebesgue measure and Baire category, yielding many interesting results.

Buck [1] has first initiated the study of the relationship between the convergence of a given sequence and the convergence of its subsequences. Miller [5], Miller and Orhan [6], Zeager [13] have studied this relation with respect to some new types of convergence. Later on, in [2], [3], [7], [8], [11], [12] more types of convergence of a sequence and the related summability of its subsequences were studied, using Lebesgue measure as a gauge of the size of the set of convergent subsequences. Also, similar relations between sequences and their subsequences were studied, using Baire category, by some authors, [4], [9]. In this paper we will return to ordinary convergence. Our aim is to study the relationship of a sequence and its subsequences with respect to ordinary convergence, as it has been done for other types of convergence, using Lebesgue measure and Baire category. We will prove some simple analogues of earlier results regarding statistical, uniform statistical and almost convergence, with the goal of offering some interesting new insights into a classical topic.

2. Main results

Throughout the paper, $x = \{x_n\}$ will always denote a sequence of real numbers. If a sequence $x = \{x_n\}$ is convergent, then all of its subsequences converge to the same limit. In general we can talk about L the set of limit (accumulation) points of x, and for simplicity, most of our results are stated for bounded sequences, but analogues easily hold for unbounded sequences (where L contains ∞ or $-\infty$).

Subsequences of a sequence $x = \{x_n\}$ can be naturally identified with numbers $t \in (0,1]$ written by a binary expansion with infinitely many 1's. For instance, the subsequence $\{x_{2n}\}$ is identified with t = 0,01010101…. Thus we can denote by x(t) =$\{(x(t))_n\}$ the subsequence of x corresponding to t. Given x(t), a subsequence of x, we will denote by L_t the set of limit points of x(t).

We are ready to state our first result.

Theorem 1   Suppose $x = \{x_n\}$ is a bounded sequence and L is the set of its limit points. Then the set of t ∈ (0,1] such that $L_t = L$ has Lebesgue measure 1.

Before proceeding to the proof of Theorem 1.1, we will prove the following Lemma.

Lemma 1   Suppose $x = \{x_n\}$ is a bounded sequence and L is the set of its limit points. If $l \in L$, then the

set of t ∈ (0,1] such that $l \in L_t$ has Lebesgue measure 1.

Proof of Lemma 1   Since l is a limit point of $\{x_n\}$, there exists a sequence $\{n_i : i \in N\}$ such that

$x_{n_i} \to l$. Any t ∈ (0,1], $t = 0. t_1 t_2 … t_n …$, for which $t_{n_i} = 1$ for infinitely many i clearly satisfies $l \in L_t$.

Let $\{n_{i_1}, n_{i_2}, … n_{i_k}\}$ be a fixed finite subset of $\{n_i : i \in N\}$. Observe the set of t ∈ (0,1] such that

$$t_n = \begin{cases} 1 & for \ n = n_{i_j}, \quad j = 1,2 … k \\ 0 & for \ n = n_i, \quad i \notin \{i_1, i_2 …. i_k\} \end{cases}$$

 (with $t_n = 0$ or 1 for other n). Since the probability of each digit t_n being 0 or 1 is $\frac{1}{2}$ (independently) and we are specifying all the digits $t_{n_i}, i = 1,2 …$, the outer measure of this set is 0. Likewise the set of t ∈ (0,1] with $t_{n_i} = 0, i = 1,2 …$, has measure 0.

Now since $\{n_i : i \in N\}$ has countably many finite subsets, the set of t ∈ (0,1], for which $t_{n_i} = 1$ for finitely many i is a countable union of sets of measure 0 and hence has Lebesgue measure 0. Hence, the set of t ∈ (0,1], for which $t_{n_i} = 1$ for infinitely many i has Lebesgue measure 1, and consequently the set of t ∈ (0,1] such that $l \in L_t$ has Lebesgue measure 1. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1   Since L is closed and separable, there exists a set $\{l_i : i \in N\} \subseteq L$ such that the closure of $\{l_i : i \in N\}$ is L. Let $T_i = \{t \in [0,1): l_i \in L_t\}$. From Lemma 1, $m(T_i) = 1$, for $i = 1,2 ….$, and hence $m(\cap_i T_i) = 1$. Now if $t \in \cap_i T_i$, then $\{l_i : i \in N\} \subseteq L_t$ and since L_t is closed, $L \subseteq L_t$. So $t \in \cap_i T_i$, implies that $L_t = L$. Therefore the set of t ∈ (0,1] such that $L_t = L$ has Lebesgue measure 1. □

We will now prove an analogous result regarding Baire category instead of Lebesgue measure. We use the term comeager to denote a set whose complement is meager or first Baire category [10].

Theorem 2   Suppose $x = \{x_n\}$ is a bounded sequence and L is the set of its limit points. Then the set of t ∈ (0,1] such that $L_t = L$ is comeager.

In a similar manner as with Theorem 1, first we prove a necessary lemma.

Lemma 2   Suppose $x = \{x_n\}$ is a bounded sequence and L is the set of its limit points. If $l \in L$, then the set of t $\in (0,1]$ such that $l \in L_t$ is comeager.

Proof of Lemma 2   Since l is a limit point of $\{x_n\}$, there exists a sequence $\{n_i : i \in N\}$ such that $x_{n_i} \to l$. For $m \in N$ let X_m denote the set of t $\in (0,1]$ for which there exists $i \geq m$ such that $t_{n_i} = 1$. We will show that X_m comeager for $m \in N$.

Suppose m is arbitrarily fixed. Let $t_1 t_2 \ldots t_k$ be an arbitrarily fixed finite sequence of 0's and 1's. Fix n_i such that $i \geq m$ and $n_i > k$. Observe the fixed sequence of length n_i , $t_1 t_2 \ldots t_k, 0,0 \ldots 0,1$ (with $(n_i - k - 1)$ $0's$ and 1 in the n_i'th position).   Then   every $t \in (0,1]$ that   starts   with   the   segment $t_1 t_2 \ldots t_k, 0,0 \ldots 0,1$ has $t_{n_i} = 1$ so t $\in X_m$. Now since numbers t $\in (0,1]$ that start with a fixed finite sequence of 0's and 1's (and have infinitely many 1's) represent a half-open interval (see [10]), this means that every (half-open) interval contains an interval that is a subset of X_m , so X_m is comeager.

Now since X_m is comeager for every m, $\cap_m X_m$ is comeager. But clearly $t \in \cap_m X_m$ implies that $x(t)$ contains a subsequence of $\{x_{n_i}\}$ and therefore has l as a limit point. Hence we conclude that the set of t $\in (0,1]$ such that $l \in L_t$ is comeager. □

We proceed to the proof of Theorem 2.

Proof of Theorem 2   As in the proof of Theorem 1, let $\{l_i : i \in N\} \subseteq L$ be such that the closure of $\{l_i : i \in N\}$ is L and let $T_i = \{t \in (0,1] : l_i \in L_t\}$. From Lemma 2 we know that T_i is comeager for each i. Hence $\cap_i T_i$ is comeager.   As earlier, $t \in \cap_i T_i$, implies that $L_t = L$. Therefore the set of t $\in (0,1]$ such that $L_t = L$ is comeager. □

We remark that the above lemmas and theorems also hold for unbounded sequences $\{x_n\}$, treating ∞, -∞ as ordinary limit points and as element(s) of L. The only reason for stating the results for bounded sequences was esthetic, as the proofs are slightly more elegant.

3.   Uniformly large subsequences

In the previous section we have seen that if l is a limit point of a sequence $x = \{x_n\}$, then almost all of its subsequences have l as a limit point in the sense of both measure and category. Now we will examine a specific situation: If l is a limit point of a sequence $x = \{x_n\}$, how many of its subsequences have a "large part" that converges to l ?

We introduce the following terminology. We will say that $X, X \subseteq N$ is uniformly large in N, if $\forall m \, \exists n$ such that $\{n + 1, n + 2, \ldots . n + m\} \subseteq X$. We remark that if X is uniformly large in N, then the upper uniform density of X is 1 (see [11]). Additionally we say that a subsequence of $x = \{x_n\}$, $\{x_{n_i}\}$ is uniformly large in x, if $\{n_i : i \in N\}$ is uniformly large in N.

We have the following theorem.

Theorem 3   Suppose $x = \{x_n\}$ is a sequence and l is one of its limit points (l finite or infinite). Then the set of t $\in (0,1]$ such that a uniformly large subsequence of $x(t)$ converges to l is comeager.

Proof of Theorem 3   Since l is a limit point of $\{x_n\}$, we can fix a sequence $\{n_i : i \in N\}$ such that $x_{n_i} \to l$. Suppose m is arbitrarily fixed.
Now suppose that $t_1 t_2 \ldots t_k$ be an arbitrarily fixed finite sequence of 0's and 1's. Fix (the smallest) n_i such that $n_{i+1} > k$. Extend $t_1 t_2 \ldots t_k$ to the finite sequence of length n_{i+m}
   $t^* = t_1 t_2 \ldots t_k 00..0100 \ldots 0100 \ldots 100 \ldots \ldots .100 \ldots 1$
that starts with $t_1 t_2 \ldots t_k$ and afterwards has 1's in positions $n_{i+1}, n_{i+2} \ldots n_{i+m}$ and 0's in between.
Then for every t $\in (0,1]$ that starts with t^* , $x(t)$ contains $x_{n_{i+1}}, x_{n_{i+2}}, \ldots x_{n_{i+m}}$ as consecutive terms

i.e. contains as m consecutive terms, m consecutive terms from $\{x_{n_i}\}$. So every $t_1 t_2 \ldots t_k$ can be extended to a sequence t^* such that every $t \in (0,1]$ that starts with t^* contains as m consecutive terms, m consecutive terms from $\{x_{n_i}\}$.

By an argument identical to the one in the proof of Lemma 2, the set of $t \in (0,1]$ such that $x(t)$ has as m consecutive terms, m consecutive terms from $\{x_{n_i}\}$ is comeager. Let Y_m denote this set.

Now we can conclude that $\cap_m Y_m$ is comeager. Now if $t \in \cap_m Y_m$, $x(t)$ contains arbitrarily long consecutive stretches from x_{n_i} and $x_{n_i} \to l$ so $x(t)$ has a uniformly large subsequence converging to l. This completes the proof. □

We can also prove a stronger theorem that is a consequence of Theorem 3.

Theorem 4   Suppose $x = \{x_n\}$ is a bounded sequence and L is the set of its limit points. Then the set of $t \in (0,1]$ such that for every $l \in L$ there is a uniformly large subsequence of $x(t)$ converging to l, is comeager.

Proof of Theorem 4   As earlier, let $\{l_i : i \in N\} \subseteq L$ be such that the closure of $\{l_i : i \in N\}$ is L and let U_i denote the set of $t \in (0,1]$ such that a uniformly large subsequence of $x(t)$ converges to l_i.   From Theorem 3 we know that U_i is comeager for each i.   Hence $\cap_i U_i$ is comeager.

Suppose $t \in \cap_i U_i$ is fixed. Now suppose $l \in L$ is arbitrarily fixed. There is a subsequence of $\{l_i : i \in N\}$ that converges to l.   For easier notation assume that $| l_i - l | < 1/i$ for all i (take out a subsequence and rename it). As a consequence of Theorem 3, since $t \in \cap_i U_i$, for each i we can fix m_i consecutive terms of $x(t)$ that are in the interval $(l_i - \frac{1}{i}, l_i + \frac{1}{i})$ with $m_i \to \infty$.   The union of these consecutive stretches of lengths m_i then clearly makes up a uniformly large subsequence of $x(t)$ that converges to l. Hence for every $t \in \cap_i U_i$, for every $l \in L$ there is a uniformly large subsequence of $x(t)$ converging to l. We conclude that $\cap_i U_i$ is the set of $t \in (0,1]$ such that for every $l \in L$ there is a uniformly large subsequence of $x(t)$ converging to l.   Since $\cap_i U_i$ is comeager, the theorem is proved. □

We remark that we can prove the above theorem for unbounded sequences as well with some slight modifications to the proof.

However, when it comes to measure, the situation is different. Depending on the structure of the sequence x either almost all or almost none of its subsequences have a uniformly large part converging to a fixed limit point of x.

Theorem 5 Suppose $x = \{x_n\}$ is a sequence and l is one of its limit points (l finite or infinite). Then the set of $t \in (0,1]$ such that a uniformly large subsequence of $x(t)$ converges to l has Lebesgue measure 1 or 0 (both can occur).

Proof of Theorem 5   Suppose l is finite (if l is infinite the proof can be easily modified). Let X_l denote the set of $t \in (0,1]$ such that a uniformly large subsequence of $x(t)$ converges to l.

First we verify that X_l is measurable. Clearly

$$X_l = \cap_i \cup_{m>i} \cup_n \left\{ t \in (0,1] : | \left(x(t) \right)_{n+j} - l | < \frac{1}{i} \text{ for } j = 1,2, \ldots m \right\}$$

($m > i$ is put to insure arbitrarily long stretches). Then since the set $\{ t \in (0,1] : | \left(x(t) \right)_{n+j} - l | < \frac{1}{i}$ for $j = 1,2, \ldots m\}$ differs from an open set by countably many elements (see [11]), X_l is measurable.

Also, X_l is a tail set (i.e. if $t \in X_l$ and t' differs from t in finitely many digits, then $t' \in X_l$). Then X_l must have Lebesgue measure 1 or 0 or be nonmeasurable (see [10]). Since X_l is measurable, we conclude that it has Lebesgue measure 1 or 0.

To show that both values occur we give the following examples:

Let x be the alternating sequence of 0's and 1's i.e. 0,1,0,1,0,1... Suppose n is arbitrarily fixed. For t ∈ (0,1] the probability that: $t_l = 0$ for i odd, and $t_l = 1$ for i even holds for $1 \leq i \leq 2n$ is $\frac{1}{2^{2n}}$. Likewise, the probability that:

$t_l = 0$ for i odd, and $t_l = 1$ for i even holds for 2kn+1 $\leq$ i $\leq$ 2(k+1)n is $\frac{1}{2^{2n}}$ (A)

for k=0,1,2… . Hence the probability that (A) does not hold for any k=0,1,2… would be
$\left(1 - \frac{1}{2^{2n}}\right)\left(1 - \frac{1}{2^{2n}}\right) \cdots \left(1 - \frac{1}{2^{2n}}\right) \cdots = 0.$

Therefore, the probability that (A) holds for at least one k is 1, i.e. the measure of the set of t ∈ (0,1] for which there exist a k such that (A) holds is 1. Clearly for each such t, x(t) contains n consecutive terms equal to1. Hence the set of t ∈ (0,1], for which x(t) contains n consecutive terms equal to1, has measure 1, for n arbitrary. By taking the intersection of these sets for all n we see that the set of t ∈ (0,1] such that a uniformly large subsequence of $x(t)$ converges to 1 has Lebesgue measure 1. The argument for 0 is analogous.

Now observe the sequence x given by 0,1,0,0,1,0,0,0,1….., i.e. with n 0's followed by a single 1, n=1,2,3…. We will check that the set of t ∈ (0,1] such that a uniformly large subsequence of $x(t)$ converges to 1 has Lebesgue measure 0. We will show that the probability that x(t) contains n consecutive 1's is less than $\frac{1}{2^{n-1}}$. If x(t) contains 2 consecutive 1's and the first one of them is x_2 than it must have $t_3 = t_4 = 0$ and the probability of this is $\frac{1}{4}$, if it contains 2 consecutive 1's and the first one of them is x_5 than it must have $t_6 = t_7 = t_8 = 0$ and the probability of this is $\frac{1}{8}$ etc. so we can conclude that the probability that x(t) contains 2 consecutive 1's is less than $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots = \frac{1}{2}$. In an analogous manner (starting with x(t) contains n consecutive 1's and the first one of them is x_2 , or x_5, or…) we can obtain that the probability that x(t) contains n consecutive 1's is much less than $\frac{1}{2^{n-1}}$.

Hence the set of t ∈ (0,1] such that x(t) contains n consecutive 1's has measure less than $\frac{1}{2^{n-1}}$ (it is easily measurable). Therefore, the set of t ∈ (0,1] such that x(t) contains n consecutive 1's for all n has measure 0, proving the set of t ∈ (0,1] such that a uniformly large subsequence of $x(t)$ converges to 1 has Lebesgue measure 0. □

Finally, we can round off Theorem 5, with the following more general theorem.

Theorem 6   Suppose $x = \{x_n\}$ is a sequence and L is the set of its limit points. Then the set of t ∈ (0,1] such that for every $l \in L$ there is a uniformly large subsequence of $x(t)$ converging to l, has Lebesgue measure 1 or 0 (both can occur).

Proof of Theorem 6   Assume x is bounded (the proof can be modified for unbounded sequences). As in Theorem 4, we fix $\{l_i : i \in N\} \subseteq L$ with closure L and let U_i denote the set of of t ∈ (0,1] such that a uniformly large subsequence of $x(t)$ converges to l_i. From Theorem 5, we know that each U_i has measure 1 or 0 and from Theorem 4 that $\cap_i U_i$ is the set of t ∈ (0,1] such that for every $l \in L$ there is a uniformly large subsequence of $x(t)$ converging to l. There are two possibilities: m$(U_i) = 1$ for all i or there exists an i such that m$(U_i) = 0$.

In the first case, we conclude that the set of t ∈ (0,1] such that for every $l \in L$ there is a uniformly large subsequence of $x(t)$ converging to l, has measure 1, while in the second case it has measure 0. Also, from the examples given in the proof of Theorem 5, we see that both may occur. □

4. Conclusion

In this paper we presented some simple but insightful new theorems showing the relationship of a sequence and its subsequences with respect to ordinary convergence, using Lebesgue measure and Baire category as gauges

of size. We proved some results analogous to earlier results regarding statistical, uniform statistical and almost convergence, providing a new viewpoint on familiar notions of convergence.

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