Periodicals of Engineering and Natural Sciences ISSN 2303-4521
Vol. 8, No. 2, June 2020, pp.821-827

Pitman estimator for the parameter and reliability function of the exponential distribution

Sabah H. Hasan[1], Abbas S. Faidh Allah[1]
[1] Department of Statistics, College of Administration and Economics, University of Kirkuk

ABSTRACT

The easiest and most essential distribution in reliability studies is the exponential distribution. An important issue in this area is to seek for a good estimator to estimate the parameter and reliability function of this distribution. So, in this article, The Pitman method is used to derive an estimator for the parameter and reliability function of the exponential distribution, then a comparison was made with the usual methods such as maximum likelihood estimator and Bayes estimator through simulation studies with different sizes of samples. The results showed that the Pitman estimator has a good performance for the parameter of exponential distribution compared with other methods by relying on the statistical measure Mean square error. Also, it showed that the Bayes estimator has a good performance compared with the other methods in estimating the reliability function by relying on the statistical measure integral mean square error.

Keywords: Exponential Distribution, Reliability Function, Pitman estimator, Maximum likelihood estimator, Bayes estimator

Corresponding Author:

Sabah H. Hasan
Department of Statistics, College of Administration and Economics
University of Kirkuk, Iraq
Email: sabahsaqi@uokirkuk.edu.iq

1. Introduction

Exponential distribution is considered one of the most important and simplest distributions in reliability studies. The scholars, in the late 1940s, have started to pick the exponential distribution for describing the life pattern of electronic systems. Davis has given an extensive diversity of instances that take account of bank statement and ledger error, automatic computing machine failure, staff check errors and radar set component failure, in which the failure data have been well designated based on exponential distribution [1]. Epstein and Sobel reported the reasons for choosing the exponential distribution over the popular normal distribution. They have shown the ways for estimating the parameter as data have been singly censored [2]. Epstein as well discussed about the justification for the assumption regarding an exponential distribution [3]. The exponential distribution has been persistent in playing a vital role in lifetime studies corresponding to normal distribution in other statistics areas [4].

Regarding the estimation methods , there are many studies that dealt with this topic for several different distributions and the use of many estimation methods, for example, we refer to Sinha [5]; Freue [6]; Akahira et al [7]; Bakhuet et al [8]; Jaafar et al [9]; Jasim [10]; Kishan [11]; Al-Obedy [12] and Triana and Purwadi [13]. This research aims to using Pitman method to derive an estimator intended for the parameter and reliability function of the exponential distribution, and to make comparison with other estimation methods such as maximum likelihood and Bayes methods via the simulation.

2. Reliability function

Reliability function is defined as the probability that the component will survives over some time period t [14].

$R(t) = p_r[T > t] = \int_t^\infty f(u)du = 1 - F(t)$ (1)

Therefore, the reliability function R(t) is related to the distribution function F(t) by F(t) + R(t) = 1

Note that the reliability function has some properties, such as [15]:

i. $0 \leq R(t) \leq 1$

ii. The component is assumed to be working properly at time t=0 and no component can work forever

without failure: i.e. $R(0) = 1$ and $R(\infty) = 0$.
iii. The function R(t) is a non-increasing function of t.

3. Exponential distribution

Exponential random variables are often used to model the lifetimes of components, for reliability analysis, and for survival analysis, among others. The exponential distribution is also used in developing models of insurance risks [4][16].
When the reliability time T follows the exponential distribution with a scale parameter λ, the probability density function is defined as

$$f(t) = \begin{cases} \frac{1}{\lambda} e^{-\frac{t}{\lambda}} & , t \geq 0,\ \lambda > 0 \\ 0 & , t < 0 \end{cases} \qquad (2)$$

The cumulative distribution function is:

$$F(t) = 1 - e^{-\frac{t}{\lambda}} \qquad (3)$$

The reliability function is given by:

$$R(t) = e^{-\frac{t}{\lambda}} \qquad (4)$$

Where the mean and variance of the exponential distribution are λ and λ^2 , respectively.

4. Parameter estimation

This section deals with three methods for finding estimators for the parameter and reliability function of exponential distribution, which are Pitman method, maximum likelihood method and Bayes method. But here the focus is on the Pitman method.

4.1. Pitman method

Let $t_1, t_2, \ldots, t_n$ be a random sample of n observations from a population whose p.d.f is $f(t;\lambda)$; where $\lambda > 0$ is a scale parameter and $t_i > 0$. If $\hat{\lambda} = h(t_1, t_2, \ldots, t_n)$ is the estimator of the scale parameter λ, then $\hat{\lambda}$ should be as follows:

$$\hat{\lambda} = \frac{\int_0^\infty \frac{1}{\lambda^2} L(t_1, t_2, \ldots, t_n; \lambda)\, d\lambda}{\int_0^\infty \frac{1}{\lambda^3} L(t_1, t_2, \ldots, t_n; \lambda)\, d\lambda} \qquad (5)$$

The Pitman estimator $\hat{\lambda}$ is also a function of sufficient statistic. If the estimator $\hat{\lambda} = h(t_1, t_2, \ldots, t_n)$ is expressed as:
$\hat{\lambda}_1 = h_1(ct_1, ct_2, \ldots ct_n) = c\, h_1(t_1, t_2, \ldots, t_n) = c\, h(t_1, t_2, \ldots, t_n) = c\hat{\lambda}$ .Then $\hat{\lambda}$ is scale invariant [17].
Now based on the above method, the likelihood function of exponential distribution is:

$$L(t_1, t_2, \ldots, t_n; \lambda) = \frac{1}{\lambda^n} e^{-\frac{\sum_{i=1}^n t_i}{\lambda}}$$

Therefore, the Pitman estimator $\hat{\lambda}_{Pitman}$ of the scale parameter λ will be as follows:

$$\hat{\lambda}_{Pitman} = \frac{\int_0^\infty \frac{1}{\lambda^{n+2}} e^{-\frac{\sum_{i=1}^n t_i}{\lambda}} d\lambda}{\int_0^\infty \frac{1}{\lambda^{n+3}} e^{-\frac{\sum_{i=1}^n t_i}{\lambda}} d\lambda}$$

Let $y = \frac{\sum_{i=1}^n t_i}{\lambda}$, $\Rightarrow \lambda = \frac{\sum_{i=1}^n t_i}{y}$, $\Rightarrow d\lambda = -\frac{\sum_{i=1}^n t_i}{y^2} dy$, then we get:

$$\hat{\lambda}_{Pitman} = \frac{\frac{1}{(\sum_{i=1}^n t_i)^{n+1}} \int_0^\infty y^n e^{-y} dy}{\frac{1}{(\sum_{i=1}^n t_i)^{n+2}} \int_0^\infty y^{n+1} e^{-y} dy}$$

$$\hat{\lambda}_{Pitman} = \frac{\frac{\Gamma(n+1)}{(\sum_{i=1}^n t_i)^{n+1}}}{\frac{\Gamma(n+2)}{(\sum_{i=1}^n t_i)^{n+2}}}$$

After simplification, we get a Pitman estimator for the parameter λ as follows:

$$\hat{\lambda}_{Pitman} = \frac{\sum_{i=1}^n t_i}{n+1} \qquad (6)$$

Therefore, using this estimator, the Pitman Reliability function of the exponential distribution will be given by:

$$\hat{R}(t)_{Pitman} = e^{-\frac{t}{\hat{\lambda}_{Pitman}}} \qquad (7)$$

4.2. Maximum likelihood method

Let $t_1,t_2,\ldots,t_n$ be a random sample of size n drawn from a population with probability density function given in equation(2), then after applying the steps of the maximum likelihood estimation (MLE) method to the exponential distribution, MLE estimator which denoted by $\hat{\lambda}_{MLE}$ will be[18]:

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} t_i}{n} = \bar{t} \qquad (8)$$

By using invariant property of MLE, then the MLE of reliability function is [19]:

$$\hat{R}(t)_{MLE} = e^{-\frac{t}{\hat{\lambda}_{MLE}}} \qquad (9)$$

4.3. Bayes method

Assume that the parameter λ has non-informative prior density defined as using Jeffrey's information π(λ) given by:

$$\pi(\lambda) \propto \sqrt{I(\lambda)}$$

Where I(λ) is Fisher information, Bayes estimator under square error loss function is mean of the posterior density function of λ given by [20]:

$$\hat{\lambda}_{Bayes} = \frac{\sum_{i=1}^{n} t_i}{n-1} \qquad (10)$$

Then Bayes estimator of the Reliability function will be:

$$\hat{R}(t)_{Bayes} = \left(\frac{\sum_{i=1}^{n} t_i}{t+\sum_{i=1}^{n} t_i}\right)^n \qquad (11)$$

5. Simulation

For the purpose of making comparison among estimation methods to estimate the parameter and reliability function of the exponential distribution, simulations were performed using the Matlab program (Matlab R2019a) with the following steps:

1) Set default values for the parameter (λ= 0.5, 1, 1.5 and 2.5), samples were taken in sizes (n = 20), (n = 50) and (n = 90, 120) representing small, moderate and large samples respectively. Samples are replicated (r=1000) times. Finally setting times of estimating reliability function (t_i= 0.1, 0.2, 0.3, 0.4, 0.5).
2) Exponential data generation through a function available in the Matlab program: x = random ('exp', λ, [n, 1]).
3) The parameter and reliability function are estimated for the exponential distribution according to the equations (6, 7) for Pitman method, equations (8,9) for the maximum likelihood method, and equations (10,11) for Bayes method.
4) The methods are compared according to the following criteria:

Regarding the comparison of parameter estimators, the comparison was made through mean squares error (MSE) which is:

$$MSE(\hat{\lambda}) = \frac{\sum_{i=1}^{r}(\hat{\lambda}_i-\lambda)^2}{r}$$

Whereas, the reliability function estimators are compared by integral mean squares error (IMSE), which is defined as distance between the estimate value of the reliability function and actual value of reliability function that is given as:

$$IMSE\,[\hat{R}(t)] = \frac{1}{r}\sum_{j=1}^{r}\left\{\frac{1}{n_t}\sum_{i=1}^{n_t}[\hat{R}_j(t_i) - R(t_i)]^2\right\} = \frac{1}{n_t}\sum_{i=1}^{n_t} MSE\{\hat{R}_j(t_i)\} \quad , \quad j=1,2,\ldots,r$$

Where: r is number of replications; n_t is the random limits of (t_i)

6. Results and discussion

The results of simulation are summarized in the tables 1,2,3,4,5, and 6. From table 1 we notice that the values of the estimators close to the default values for λ as the sample size increases. Also from table 1 it was found that for all sample sizes (n=20, 50, 90 and 120) , Pitman estimator resulted a good performance compared to the other estimators for estimating the parameter (λ) since it achieved the smallest MSE, mentioned in bold font, followed by the MLE estimator. From Tables 2,3,4 and 5, we notice that when the sample size increased, the estimated values of the reliability function are closer to the actual values of the reliability function R(t). We also observed that the estimated value of the reliability function decreases by increasing the time (t_i), and this indicates that there is an opposite relationship between failure times and the reliability function. From Table 6, it was found that Bayes method estimator got a good performance for estimating the reliability function compared to the other estimators because it achieved the smallest IMSE, mentioned in bold font, for all real λ and samples followed by the MLE estimator.

Table 1. Parameter estimation and mean square error for the estimation methods for different parameter values and sample sizes

λ	n	Pitman		Bayes		MLE	
		$\hat{\lambda}$	Mse($\hat{\lambda}$)	$\hat{\lambda}$	Mse($\hat{\lambda}$)	$\hat{\lambda}$	Mse($\hat{\lambda}$)
λ=0.5	20	0.4752	**0.0113**	0.5253	0.0137	0.4990	0.0118
	50	0.4892	**0.0049**	0.5092	0.0053	0.4990	0.0050
	90	0.4964	**0.0029**	0.5075	0.0031	0.5019	0.0030
	120	0.4974	**0.0021**	0.5058	0.0022	0.5015	0.0022
λ=1	20	0.9646	**0.0433**	1.0661	0.0557	1.0128	0.0465
	50	0.9830	**0.0196**	1.0232	0.0215	1.0027	0.0201
	90	0.9960	**0.0108**	1.0184	0.0116	1.0071	0.0111
	120	0.9949	**0.0086**	1.0117	0.0090	1.0032	0.0087
λ=1.5	20	1.4412	**0.1100**	1.5929	0.1388	1.5133	0.1177
	50	1.4739	**0.0448**	1.5341	0.0490	1.5034	0.0459
	90	1.4905	**0.0239**	1.5240	0.0255	1.5070	0.0244
	120	1.4906	**0.0182**	1.5157	0.0190	1.5031	0.0185
λ=2.5	20	2.4017	**0.3071**	2.6545	0.3872	2.5218	0.3284
	50	2.4742	**0.1212**	2.5752	0.1362	2.5237	0.1259
	90	2.4846	**0.0686**	2.5405	0.0731	2.5122	0.0701
	120	2.4906	**0.0545**	2.5325	0.0573	2.5114	0.0555

Table 2. Estimate of the reliability function for different sample sizes when λ=0.5

n	t_i	R(t)	M.L.E	Bayes	Pitman
20	0.1	0.8187	0.8112	0.8121	0.8028
	0.2	0.6703	0.6596	0.6625	0.6461
	0.3	0.5488	0.5375	0.5427	0.5213
	0.4	0.4493	0.4390	0.4463	0.4217
	0.5	0.3679	0.3592	0.3668	0.3418
50	0.1	0.8187	0.8163	0.8166	0.8130
	0.2	0.6703	0.6668	0.6679	0.6615
	0.3	0.5488	0.5451	0.5472	0.5386
	0.4	0.4493	0.4460	0.4489	0.4389
	0.5	0.3679	0.3652	0.3689	0.3580
90	0.1	0.8187	0.8167	0.8169	0.8149
	0.2	0.6703	0.6673	0.6679	0.6643
	0.3	0.5488	0.5455	0.5466	0.5419
	0.4	0.4493	0.4461	0.4477	0.4422
	0.5	0.3679	0.3650	0.3671	0.3610
120	0.1	0.8187	0.8176	0.8178	0.8162
	0.2	0.6703	0.6687	0.6692	0.6665
	0.3	0.5488	0.5471	0.5479	0.5444
	0.4	0.4493	0.4477	0.4489	0.4447
	0.5	0.3679	0.3665	0.3681	0.3635

Table 3. Estimate of the reliability function for different sample sizes when λ=1

n	t_i	R(t)	MLE	Bayes	Pitman
20	0.1	0.9048	0.9001	0.9004	0.8954
	0.2	0.8187	0.8108	0.8117	0.8023
	0.3	0.7408	0.7307	0.7325	0.7194
	0.4	0.6703	0.6589	0.6618	0.6455
	0.5	0.6065	0.5945	0.5989	0.5795
50	0.1	0.9048	0.9034	0.9035	0.9016
	0.2	0.8187	0.8163	0.8167	0.8130
	0.3	0.7408	0.7378	0.7385	0.7333
	0.4	0.6703	0.6669	0.6681	0.6616
	0.5	0.6065	0.6030	0.6046	0.5970
90	0.1	0.9048	0.9037	0.9037	0.9026
	0.2	0.8187	0.8167	0.8169	0.8149
	0.3	0.7408	0.7382	0.7386	0.7357
	0.4	0.6703	0.6673	0.6679	0.6643
	0.5	0.6065	0.6033	0.6041	0.5999
120	0.1	0.9048	0.9043	0.9043	0.9035
	0.2	0.8187	0.8178	0.8178	0.8163
	0.3	0.7408	0.7398	0.7398	0.7377
	0.4	0.6703	0.6693	0.7398	0.6666
	0.5	0.6065	0.6057	0.6057	0.6025

Table 4. Estimate of the reliability function for different sample sizes when λ=1.5

n	t_i	R(t)	MLE	Bayes	Pitman
20	0.1	0.9355	0.9316	0.9317	0.9283
	0.2	0.8752	0.8681	0.8685	0.8620
	0.3	0.8187	0.8091	0.8100	0.8006
	0.4	0.7659	0.7544	0.7559	0.7439
	0.5	0.7165	0.7035	0.7058	0.6914
50	0.1	0.9355	0.9340	0.9341	0.9328
	0.2	0.8752	0.8725	0.8727	0.8701
	0.3	0.8187	0.8151	0.8155	0.8118
	0.4	0.7659	0.7616	0.7621	0.7574
	0.5	0.7165	0.7116	0.7124	0.7068
90	0.1	0.9355	0.9349	0.9350	0.9342
	0.2	0.8752	0.8742	0.8743	0.8729
	0.3	0.8187	0.8174	0.8176	0.8156
	0.4	0.7659	0.7643	0.7646	0.7620
	0.5	0.7165	0.7147	0.7152	0.7121
120	0.1	0.9355	0.9350	0.9351	0.9345
	0.2	0.8752	0.8743	0.8744	0.8734
	0.3	0.8187	0.8176	0.8177	0.8162
	0.4	0.7659	0.7646	0.7648	0.7629
	0.5	0.7165	0.7150	0.7154	0.7131

Table 5. Estimate of the reliability function for different sample sizes when λ=2.5

n	t_i	R(t)	MLE	Bayes	Pitman

20	0.1	0.9608	0.9593	0.9593	0.9573
	0.2	0.9231	0.9203	0.9204	0.9165
	0.3	0.8869	0.8830	0.8833	0.8775
	0.4	0.8521	0.8472	0.8478	0.8403
	0.5	0.8187	0.8130	0.8139	0.8047
50	0.1	0.9608	0.9601	0.9601	0.9593
	0.2	0.9231	0.9218	0.9219	0.9203
	0.3	0.8869	0.8851	0.8853	0.8830
	0.4	0.8521	0.8499	0.8501	0.8471
	0.5	0.8187	0.8161	0.8164	0.8128
90	0.1	0.9608	0.9603	0.9603	0.9599
	0.2	0.9231	0.9222	0.9223	0.9214
	0.3	0.8869	0.8857	0.8858	0.8845
	0.4	0.8521	0.8506	0.8507	0.8491
	0.5	0.8187	0.8169	0.8171	0.8151
120	0.1	0.9608	0.9605	0.9605	0.9602
	0.2	0.9231	0.9226	0.9226	0.9220
	0.3	0.8869	0.8862	0.8863	0.8853
	0.4	0.8521	0.8513	0.8514	0.8501
	0.5	0.8187	0.8177	0.8178	0.8163

Table 6. IMSE of estimation methods of reliability function for different sample sizes and parameter values

λ	n	MLE	Bayes	Pitman
λ=0.5	20	0.0050	**0.0047**	0.0056
	50	0.0018	**0.0017**	0.0019
	90	0.0011	**0.0010**	0.0011
	120	7.2353e-04	**7.1681e-04**	7.4082e-04
λ=1	20	0.0029	**0.0027**	0.0034
	50	0.0011	**0.0010**	0.0011
	90	5.4237e-04	**5.3653e-04**	5.6839e-04
	120	4.4203e-04	**4.3905e-04**	4.5448e-04
λ=1.5	20	0.0018	**0.0017**	0.0022
	50	6.0685e-04	**5.9825e-04**	6.6052e-04
	90	3.2081e-04	**3.1860e-04**	3.3459e-04
	120	2.7131e-04	**2.6991e-04**	2.7973e-04
λ=2.5	20	7.2060e-04	**7.0525e-04**	8.5850e-04
	50	2.6235e-04	**2.6012e-04**	2.8471e-04
	90	1.5738e-04	**1.5661e-04**	1.6511e-04
	120	1.0619e-04	**1.0582e-04**	1.0993e-04

7. Conclusion

In this paper we have derived an estimator for the parameter and the reliability function of the exponential distribution by using Pitman method, then the comparison was made with the other estimation methods such as maximum likelihood estimation and Bayes estimation methods via simulation. It was found that Pitman estimator resulted a good performance compared with the other methods in estimating the parameter of exponential distribution, while Bays estimator has a good performance for the reliability function of the same distribution.

References

[1] D. J. Davis, “An Analysis of Some Failure Data.” Journal of the American Statistical Association, vol. 47, pp. 113-150, 1952.
[2] B. Epstein and M. Sobel, “Life Testing. “Journal of the American Statistical Association, vol. 48, pp. 486-502, 1953.
[3] B. Epstein, “The Exponential Distribution and Its Role in Life Testing. “Industrial Quality Control, vol.15, pp. 2-7, 1958.
[4] E. T. Lee and J. W. Wang,” Statistical Methods for Survival Data Analysis.”, Third Edition, John Wiley & Sons. 2003.
[5] S. K. Sinha,” Bayes Estimation of Reliability Function of Normal Distribution.”, IEEE Trans, vol.34, no. 2, pp. 360-364, 1985.
[6] G. C. Freue, “The Pitman Estimator of the Cauchy Location parameter.”, Journal of Statistical Planning and Inference”, vol. 137, no. 6, pp. 1900-1913, 2007.
[7] M. Akahira, N. Ohyauchi, and K. Takeuchi,” On the Pitman Estimator for a Family of No Regular Distributions.”, METRON-International Journal of Statistics., vol. 65, no. 1, pp. 113-127, 2007.
[8] A. K. Bakhuet, I. A. Hason, and A. N. Hussein, “Simulate Five Estimation Methods for Parameter and Reliability Function of Exponential Distribution.” Journal of Economics and Administrative Sciences, Vol.15, no.53, pp. 200-222, 2009.
[9] S. M. Jaafar, B. I. Abdulwahhab and I. A. Hassom, “Best Estimation for the Reliability of 2- Parameter Weibull Distribution”, Baghdad Journal of Science, vol. 6, no. 4, pp. 705-710, 2009.
[10] W. A. Jasim,” Bayes Estimator of One Parameter Gamma Distribution Under Quadratic and LINEX Loss Function.”, Iraqi Journal of Statistical Science, vol. 9, no. 16, pp. 13-28, 2010.
[11] R. Kishan, “Comparison Between MLE and Bayes Estimation of Scale Parameter of Generalized Gamma Distribution with Known Shape Parameter Under Squared Error Loss Function.”, Journal of Reliability and Statistical Studies, vol. 7, no. 1, pp. 43-50, 2014.
[12] J. A. Al-Obedy, “Comparison Bayes Estimators of Reliability in the Exponential Distribution. “, Journal of Economics and Administrative Sciences, Vol.24, no.104, pp. 1-36, 2018.
[13] Y. Triana and J. Purwadi,” Exponential Distribution Parameter Estimation with Bayesian SELF Method in Survival Analysis.”, Journal of Physics: Conference Series 1373,012050, 1-7, DOI:10.1088/1742-6596/1373/1/012050, 2019.
[14] A. H. Al-Nasser, “An Introduction to Statistical Reliability “, ITHRAA Publishing and Distribution, 2009.
[15] W. Q. Meeker and L. U. Escobar,” Statistical Methods for Reliability Data.” John Wiley & Sons., 1998.
[16] K. M. Ramachandran and C. P. Tsokos, “Mathematical Statistics with Applications in R.” Second Edition, Elsevier, 2015.
[17] K. C. Bhuyan, “Probability Distribution Theory & Statistical Inference.”, New Central Book Agency Pvt. Ltd, New Delhi. 2010.
[18] R. V. Hogg, E. A. Tanis and D. L. Zimmerman, “Probability and Statistical Inference. “Ninth Edition, Pearson Education, Inc. 2015.
[19] B. M. Kurkjian, G. Q. Strong and M. J. Karson,” Reliability estimation for the exponential distribution.”, Communications in Statistics - Simulation and Computation, vol. 16, no. 3, pp. 835-853, 1987.
[20] H. S. Al-Kutubi, and N. A. Ibrahim, “Bayes Estimator for Exponential Distribution with Extension of Jeffery Prior Information.,” Malaysian Journal of Mathematical Sciences, vol. 3, no. 2, pp. 297-313, 2009.