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Properties of the fundamental splines of the high order

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Abstract

In this paper, the properties of the fundamental splines of high orders are numerically studied. From the point of view of the interpolation problem, the fundamental spline is a nodal function generated by a family of integer translations of the corresponding basic spline. We have established that with increasing order fundamental splines tend to the sampling function $\operatorname{sinc}(\pi x)$. Analogous assertions were obtained earlier for nodal functions on the basis of other systems of shifts. The behavior of the coefficients of basic splines is studied. With the help of calculations, it is shown that for n>2 there is a sign-reversal and a monotone decreasing modulo.

Keywords: Basic spline, Fundamental spline, Interpolation, Node function, System of integer translates, Sampling function

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1. Introduction

Spline functions and their various modifications play an important role in computational mathematics and computer processing of information [1, Ch. 4], [2], [3]. The most frequently used are cubic splines, for working with which simple and effective algorithms are developed. In the case of splines of higher orders, constructions called basic and fundamental splines are used [1, Ch. 4]. This method, despite its theoretical simplicity, in practical application leads to considerable difficulties of a computational nature. Therefore, at present, some properties of basic and fundamental splines of high orders are insufficiently studied.

The basic spline interpolation is closely related to the interpolation problem for integer-shift systems. The most popular and well-studied at the present time is the system of integer shifts of the Gaussian function [4, 5, 6]. Also recently a number of papers have appeared devoted to the study of the interpolation properties of the shifts of the so-called Lorentz function [7, 8, 9] and its Fourier transform, that led to one of generalizations of basic splines [10]. In the works mentioned, questions on the stability of the analysis and synthesis procedure with the help of these systems, the behavior of the coefficients of nodal functions, and some important limit relationships were obtained. These questions are open to high order splines, the study of which is hampered by the absence of simple analytical expressions for them. Therefore, in this paper, a numerical construction of basic and fundamental splines of high orders is carried out, as well as the study of regularities in the behavior of the coefficients of fundamental splines.

2. Basic and fundamental splines

For the construction of basic splines, the characteristic function of the segment $\chi_{[a,b]}(x)$ is required,

$$\chi_{[a,b]}(x) = \begin{cases} 1, x \in [a,b], \\ 0, x \notin [a,b]. \end{cases}$$

together with the convolution operation for functions defined on the whole number axis.

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t) \cdot g(t) dt.$$

Centered basic splines of arbitrary order are given by the following relations [1, Ch. 4]

$$B_1(x) = \chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x) \ , \tag{1}$$

$$B_n(x) = (B_{n-1} * B_1)(x) = \int_{-\infty}^{+\infty} B_{n-1}(x-t) \cdot B_1(t) dt \tag{2}$$

We give some of their properties.

1. $B_n(x) > 0$ for all $-\frac{n}{2} < x < \frac{n}{2}$, $B_n(x) = 0$ when $|x| \geq \frac{n}{2}, n > 1$.

2. For all $x \in IR$ the equality is true

$$\sum_{k=-\infty}^{\infty} B_n(x-k) = 1.$$

3. We have the relation

$$B_n'(x) = B_{n-1}\left(x + \frac{1}{2}\right) - B_{n-1}\left(x - \frac{1}{2}\right).$$

4. B-splines $B_n(x)$ and $B_{n-1}(x)$ are connected by the identity

$$B_n(x) = \frac{n+2x}{2(n-1)} B_{n-1}\left(x + \frac{1}{2}\right) + \frac{n-2x}{2(n-1)} B_{n-1}\left(x - \frac{1}{2}\right) \tag{3}$$

5. The spline is symmetric

$$B_n(-x) = B_n(x) \quad .$$

6. The following formula is valid

$$B_n(x) = \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k C_n^k (x + \frac{n}{2} - k)_+^{n-1} \tag{4}$$

where C_n^k is the binomial coefficient, $x_+ = \dfrac{x + |x|}{2}$.

Fundamental splines play an important role in interpolation problems. These are functions that are a linear combination of basic ones

$$F_n(x) = \sum_{k=-\infty}^{\infty} d_k \cdot B_n(x-k) \tag{5}$$

and satisfying the conditions $F_n(m)=\delta_{0m}, m\in Z$. The construction of a fundamental spline reduces to the problem of interpolation by systems of integer shifts.

For the construction it is actually required to solve an infinite system of linear equations with an infinite number of unknowns d_k

$$\sum_{k=-\infty}^{\infty} d_k B_n(m-k)=\delta_{0m}, m\in Z \qquad (6)$$

This formula is a convolution type system. Equations of convolution type in the continuous case are usually solved with the help of the Fourier transform; in the discrete infinite case, the Fourier series is used for this, and in the finite-dimensional case the discrete Fourier transform is applied.
Trigonometric series [1, Ch. 4], [11, Ch. 1]

$$D(t)=\sum_{k=-\infty}^{\infty} d_k e^{ikt}$$

is called a symbol or a mask of an infinite sequence $\{d_k\}_{k\in Z}$.
Let $\Phi(t)$ be a mask constructed from the values of the function at integral nodes, that is,

$$\Phi(t)=\sum_{j=-\infty}^{\infty} B_n(j)e^{ijt} \qquad (7)$$

. The following statement is valid, which can easily be verified directly [4]: if the trigonometric series $D(t)$ and $\Phi(t)$ converge absolutely, then for them the equality is true

$$D(t)\cdot\Phi(t)=1 \qquad (8)$$

Thus, the infinite system of equations (6) in this case reduces to the functional equality (8). Then, to determine the coefficients, it is necessary to expand to the Fourier series the function $1/\Phi(t)$. This circumstance makes it possible not to solve directly the system of equations (6).

3. Construction of basic splines

The construction of basic splines by definition with convolutions (1) – (2) seems rather cumbersome. In addition, the calculation of the corresponding integrals by quadrature formulas introduces an additional and essential error.
As calculations have shown, the calculation of basic splines according to formula (4) allows reliable construction of splines only up to the 70th order. The most effective and convenient for practical application is the recurrence formula (3). With the aid of (3), it is possible to find stably spline bases over 1000th order. Below in Fig. 1-2 shows graphs of basic splines of various orders obtained by the above method.

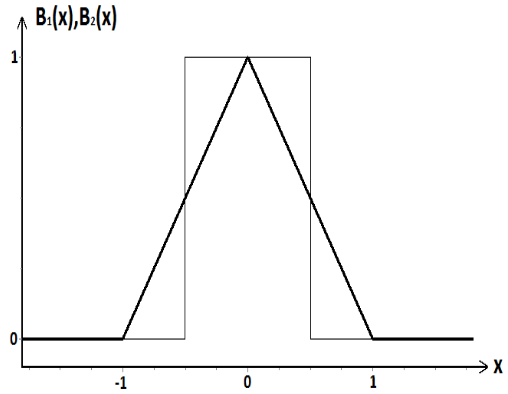

Fig. 1. Basic splines of the 1st (the usual line) and 2nd order (a thick line)

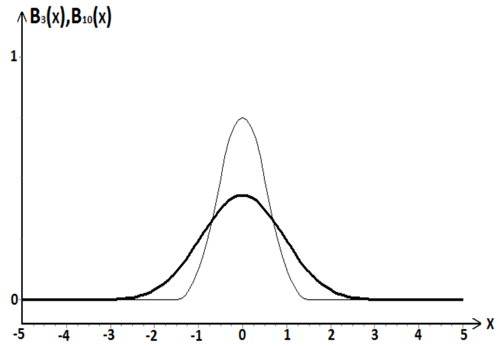

Fig. 2. Basic splines of the 3rd (the usual line) and the 10th order (the thick line)

In Fig. 3 shows an enlarged graph of the basic spline of the 1000th order. It should be noted that although the support of $B_{1000}(x)$ is a [–500,500] segment, but even with $|x|>50$ a spline is practically indistinguishable from zero.

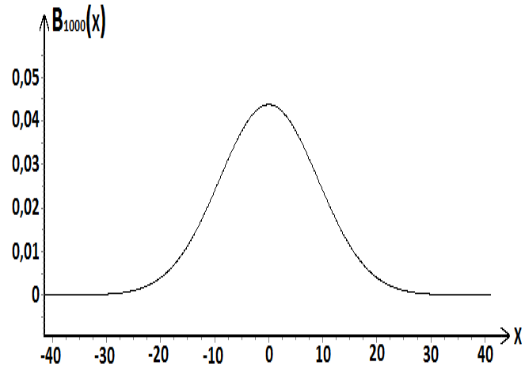

Fig. 3. Basic spline of the 1000th order

4. Construction of fundamental splines

To construct a fundamental spline $F_n(x)$ of order n, we apply the following algorithm, similar to that used in [6].

1. We construct a basic spline $B_n(x)$, described in the above way, and calculate its values at integer points $B_n(k)$. Since $B_n(-x)=B_n(x)$ we can only limit ourselves to positive values k. In addition, outside the segment $\left[-\frac{n}{2},\frac{n}{2}\right]$ the base spline is zero, hence, it remains to find only the values of $B_n(k)$ for the values of k from zero to the integer part of the number $\frac{n}{2}$.

2.We construct an auxiliary series (7) for the fundamental spline. Formula (7) can be considerably simplified. First, the summation, on the basis of what was said above, does not need to be carried out in infinite limits. Secondly, it is convenient to go over to real quantities. We denote by $N(n)$ the integer part of $\frac{n}{2}$. Then (7) is transformed to the form

$$\Phi(t)=B_n(0)+2\sum_{j=1}^{N(n)}B_n(j)\cos(jt). \qquad (9)$$

We shall use formula (9) to calculate the mask $\Phi(t)$.

3. Further, according to (8), in order to find the coefficients d_k of the fundamental spline, we must expand the function $D(t) = 1/\Phi(t)$ in a Fourier series. We use the discrete Fourier transform. To do this, we divide the segment $[0,\pi]$ into M parts with the same step $h = \pi / M$. Then we can write the following approximate expression for the Fourier coefficients of the function $D(t)$ [6]

$$d_k \approx \frac{1}{2M}\left(D(0) + 2\sum_{m=1}^{M-1} D(mh)\cos(mkh) + (-1)^k D(\pi) \right) \qquad (10)$$

4. The last stage consists in constructing the fundamental spline itself in the form of a linear combination of basis ones, that is, by formula (5).

4. Analysis of the properties of fundamental splines of high orders

First, we consider the behavior of fundamental splines graphically. Splines of the first and second orders, as is well known, coincide with the corresponding basic splines. Note that, although it decreases rapidly, it is not finite. This can be seen by direct calculation. Nevertheless, from a practical point of view, its values can be neglected in regions sufficiently far from the origin.
When considering splines of high orders, it is interesting to compare them with the so-called sample function, which is important in digital signal processing

$$\operatorname{sinc}(\pi x) = \frac{\sin(\pi x)}{\pi x}. \qquad (11)$$

The fact is that a number of new facts have been established in recent papers [5, 6, 8, 10] concerning the nodal functions generated by the shifts of the Gaussian and Lorentz functions. One of them is that with increasing width of the Gaussian functions (the same for the Lorentz function), the corresponding nodal functions tend to $\operatorname{sinc}(\pi x)$. With increasing order, the basic and fundamental splines also tend to expand, so it will be interesting to check whether they also tend to the function (11).

We give some graphs of fundamental splines of various orders (a thick line), overlapping the graph of the function (11) (the usual line). In Fig. 4 it is shown a fundamental spline of the 5th order. Near the middle there is a similarity between it and the sample function, along the edges there are significant discrepancies.

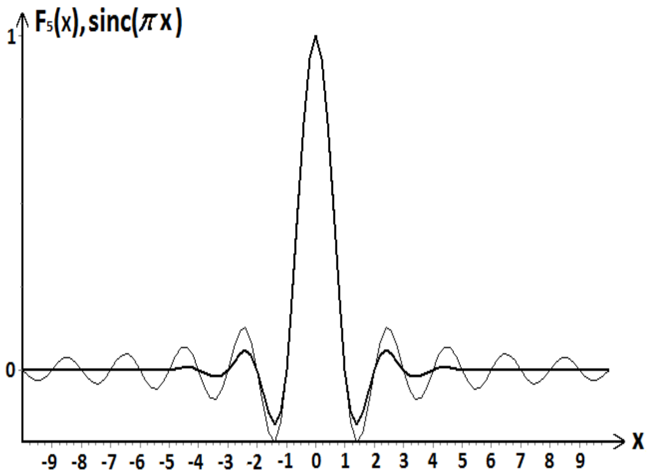

Fig. 4. The fundamental spline of the 5th order (bold line) and the sample function (the usual line)

In Fig. 5 it is shown a fundamental spline of the 15th order. In the middle, with the sample function, they became visually indistinguishable, the difference along the edges is also noticeably smaller. The graph of the fundamental spline of the 30th order (Figure 6) confirms the planned trended.

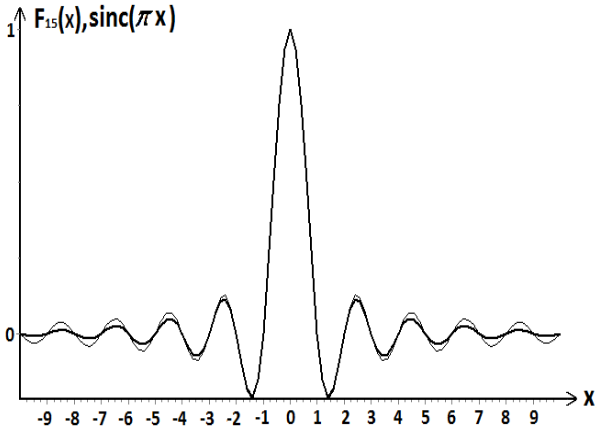

Fig. 5. The fundamental spline of the 15th order (bold line) and the sample function (the usual line)

Thus, as follows from the calculations, with increasing order, the fundamental spline gradually approaches the sample function. Theoretical substantiation of such statements requires the involvement of a complex mathematical apparatus. Therefore, within the framework of this paper, we confine ourselves to computational experiments.

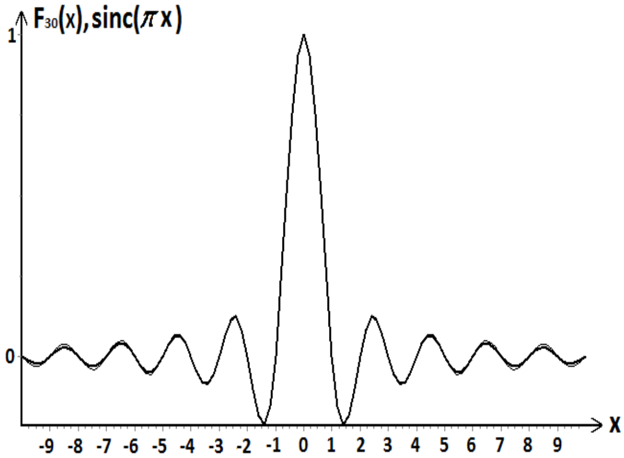

Fig. 6. The fundamental spline of the 30th order (bold line) and the sample function (the usual line)

Let us pass to the study of the regularities in the behavior of the coefficients of the fundamental spline. Table 1 shows the values of several coefficients of the fundamental spline of the 10th order. Table. 2 and 3 contain information on the coefficients of the fundamental splines of the 15th and 20th order, respectively. All significant figures are correct to within rounding.

Table 1. The values of the coefficients of the fundamental spline of the 10th order

Number, k	Value, d_k	Number, k	Value, d_k	Number, k	Value, d_k
0	9,237	5	–1,018	10	0,08469
1	–6,833	6	0,6196	11	–0,05149
2	4,409	7	–0,3768	12	0,03131
3	–2,732	8	0,2291	13	–0,01904
4	1,671	9	–0,1393	14	0,01157

Table 2. Values of the coefficients of the fundamental spline of the 15th order

Number, k	Value, d_k	Number, k	Value, d_k	Number, k	Value, d_k
0	53,37	5	–15,54	10	3,019
1	–47,80	6	11,24	11	–2,170
2	38,22	7	–8,106	12	1,560
3	–28,94	8	5,836	13	–1,121
4	21,34	9	–4,198	14	0,8061

Table 3. Values of the coefficients of the fundamental spline of the 20th order

Number, k	Value, d_k	Number, k	Value, d_k	Number, k	Value, d_k
0	365,2	5	–168,5	10	50,77
1	–347,2	6	133,7	11	–39,70
2	306,2	7	–105,4	12	31,03
3	–257,4	8	82,79	13	–24,24
4	210,1	9	–64,88	14	18,94

First of all, we note that the coefficients grow in modulus with increasing order of the spline. For a spline of the same order, as can be observed from the data in the tables, the coefficients of the fundamental splines alternate and decrease monotonically in modulus with increasing number. This fact is very useful for the convergence of interpolation series.

5. Conclusion

In this paper we have established by calculations that with increasing order fundamental splines tend to the sampling function. This fact also takes place for some another system of integer translates, therefore it can be sign of common regularity in behavior of the node function. It requires however a more detailed study.

The noted properties of coefficients of the fundamental spline are far from trivial. For example, the coefficients of the nodal function generated by the Gaussian function also alternate and monotonically decreasing, and in the case of the Lorentz function this is no longer valid.

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