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Fuzzy reliability estimation for Frechet distribution by using simulation

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ABSTRACT

The study has examined the estimation of Frechet distribution parameters with the shape parameter (α) and the scale parameter (β). Two estimation methods are used based on the maximum likelihood and Bayes methods. The life time data are fuzzy numbers. These estimations of parameters are employed to estimate the fuzzy reliability function of the distribution and to select the best estimate of the fuzzy reliability function by comparing the mean squares error (MSE) and the average absolute proportional error (MAPE). The results of simulation showed that the fuzziness is better than the real for all sample sizes and the fuzzy reliability at the estimates of the Bayes estimated is better than the maximum likelihood method. It gives the lowest average MSE and MAPE until to arrive at a minimum at sample size of n = 500.

Keywords: Fuzzy lifetime data, Fuzzy numbers, Maximum likelihood estimation, Bayes estimation

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1. Introduction

In the real world, we encounter sets of things that do not have the exact membership of their elements. So, these sets do not form sets in the usual mathematical sense of these terms. For example, if we have a class of students at a particular university, we can take a term representing "students who have a driver's license". It is natural that each element (student) in that set either has a license or does not. If the student has a driving license and the value is zero, the student cannot have a driving license. But, if we take the subset that is "students who drive very well", each student will have a certain degree of quality of driving and an element in the set is distinguished by a membership function that gives values between zero and one and here partial affiliation is allowed.

In the other hand, reliability is one of the most important and effective techniques at present to evaluate the work of systems or units. It is the function that gives the probability of any unit or vehicle working for a certain period of time without failure. Many methods and models in the theory of reliability in its conventional form assume that all the parameters of the life-time probability function are crisp. In the real world applications, it is necessary to generalize the classical statistical estimation methods of the real numbers into fuzzy numbers. This is because of the parameters of probability distribution sometimes cannot be accurately recorded due to errors of experience, personal judgment, estimation, or some unexpected situations. Then, the parameters in the life distributions are fuzzy. So, the system of reliability may become difficult to deal with the function of traditional reliability. Consequently, we can deal with a more comprehensive term than the traditional term of reliability. This is defined as the fuzzy probability of continuity of work of the vehicle or unit successfully for a certain period of time and to the degree of belonging determined according to the function of a certain membership.

The fuzzy logic began to mature by Zadeh in 1965, when he used the term "fuzzy variables" in terms of approximate or inaccurate linguistic expressions and language [1]. This is the first to establish the foundations of the theory of fuzzy sets theory. The fuzzy set is defined as a set of objects or elements that have continuous degrees of membership. They are characterized by the function of membership to each object in the set. The degree of membership is often between zero and one [2], [3]. Cheng in 1996, proposed an abbreviation of the graphical evaluation and review technique. This method was used to calculate the fuzzy reliability of warplanes for several successful flight attempts. Wu in 2004, estimated fuzzy reliability using the Bayes approach in the fuzzy environment by assuming a fuzzy treatment of fuzzy variables with previous fuzzy distributions [4]. The traditional Bayes estimation method was used to create the fuzzy estimator of reliability by including the theory of resolution identity and determine the degree of membership to any Bayes estimate of reliability. Huang and Zuo in 2006, analyzed basis reliability of fuzzy life data. Bayes method was used to estimate fuzzy reliability based on the size of a small sample by assuming a new method to determine the function of membership estimation and the reliability function of multi-parameters life distributions [5]. Abbas and Yincai in 2012, estimated the scale parameter for Frechet distribution with a parameter of a given shape. They used the maximum likelihood, weighted moments and the Bayes methods for the prior Jeffery distribution. The quadratic loss function, El-Sayyad loss function, and the linex loss function was as well done using different sample sizes based on the MSE error standard [6]. They concluded that the method of maximum likelihood is better than the Bayes method in terms of Bayes when increasing the value of α. Pak and Saraj in 2013, estimated the reliability of Rayleigh distribution distributions based on fuzzy life data by using the Bayes method to estimate the parameter and reliability function of distributions from fuzzy life data. Because the Bayes estimations cannot be given in clear formats, the researchers used approximations such as Lindely, Tierney and Kandane approximations. The results were showed that the approximations of Tierney and Kandane's give accurate estimates of the parameters [7]. So, it is recommended to use it to find Bayes estimates as well as the reliability of the Rayleigh distribution. Pak, Ali and Saraj have used maximum likelihood, Bayes estimation and moment estimations. The observations are in the form of fuzzy data in the Weibull distribution by using the Newton Rafson method and the maximization expectation method. The aim of work was to find the estimates of the maximum likelihood estimation and the approximation of Tierney and Kandane's approximation for Bayes estimates based on a quadratic loss function and a repetitive method for finding the inverse estimation [8]. The Monte Carlo simulation was employed when the sample size is small and medium. Bayes estimator is better than the estimator of the maximum likelihood and then the moment's method comes after them. If the sample size is large, the three methods will give the same estimates. Pak in 2016, used maximum likelihood estimation, Bayes and moments estimations to estimate the shape parameter for log-normal distribution when the observations are in a fuzzy data format. He has used the Newton Rafson method to find the maximum likelihood estimates and the Monte-Carlo Markov series method to find Bayes estimates for different types of prior distributions [9]. The Monte Carlo simulation was used to compare among methods. Bayes estimates based on non- informative prior distribution as well as estimations of the maximum likelihood estimation have yielded similar estimation results. In the case of the previously fully informative Bayes estimates, the least mean error squares are present.. The average error and Bayes estimators will decrease significantly if the sample size increases. Nathier and Mohammed in 2017, were estimated the parameters of the reliability for fuzzy exponential distribution with two parameters [10]. The λ parameter was estimated by the method of moments and the maximum likelihood [11]. They estimated the fuzzy reliability function and compared the results using the MSE. They found that the maximal likelihood estimator is the best and the fuzzy reliability is better when the estimator is for maximum likelihood at the fuzzy number $k = 0.3$. They concluded that the fuzzy estimate of reliability is better than the conventional estimate. In the same year, Pak used Lindley distribution with one parameter when the data is available in a fuzzy data format using the maximum likelihood estimation and Bayes Estimation by EM equations to determine the MLE of the parameter and establish confidence limits using the asymptotic normality of the maximum potential estimator [12]. Laplace, known as the Monroe Monte Carlo chains was created to find a Bayes estimator for the parameter. A previous confidence period was also obtained for the unknown parameter. Through the Monte Carlo study, the researcher concluded that Bayes estimates based on prior non-informative as well as estimates of the maximum likelihood have yielded similar estimation results. In the case of the prior information, the estimate of Bayes has the lowest average error squares.

2. Materials and method

Frechet distribution is one of the most recent probability distributions of lifetime models. This distribution was provided by Maurice Frechet. It has extensive applications in the modeling and analysis of many events such as earthquakes, earthquakes, floods, rainfalls, wind speeds, life tests, and sea currents. Statistical behavior of material properties can be used in engineering fields as well as in infant mortality modeling and modeling of special events and maintenance periods. The Frechet distribution is used to model failure rates that are commonly used in reliability biological studies, optical signal analysis and error modeling.

Drapella and Kollia were suggested an inverse named a reciprocal of Weibull on the distribution of Frechet distribution [13].
If x is a random variable with a Weibull distribution, then y = 1 / x is the inverse of the values of the random variable x. It has a probability distribution of the following probability density:

$$f(x,\alpha,\beta) = \alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \;;\; x > 0 \tag{1}$$

Here, α > 0 is the shape parameter and $\beta > 0$ is for scale Parameter
The cumulative distribution function is given as follows [16]:

$$F(x) = P(X \le x) = \int_0^x f(u)du = \exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \;;\; x > 0 \tag{2}$$

The reliability function is given as follows:

$$R(t) = 1 - F(t) = \int_t^{\infty} f(t)dt = 1 - \exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right) \tag{3}$$

The hazards function can determined by:

$$H(t) = \alpha\beta^{\alpha}t^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{t}\right)^{\alpha}\right)\left(1 - \exp\left(-\left(\frac{\beta}{t}\right)^{\alpha}\right)\right)^{-1} \tag{4}$$

The k-rank is about the point of origin K 'th with moment about origin:

$$\mathrm{EX}^k = \int_0^{\infty} \mathrm{X}^k f(x)dx = \beta^k\,\Gamma(1-\tfrac{k}{\alpha}) \;;\; k=1,2,3,\ldots \tag{5}$$

Mean and variance are computed by:

$$\mu_x = \beta\Gamma(1-\tfrac{1}{\alpha}) \tag{6}$$

$$\sigma_x^2 = \beta^2[\Gamma(1-\tfrac{2}{\alpha}) - \Gamma^2(1-\tfrac{1}{\alpha})] \tag{7}$$

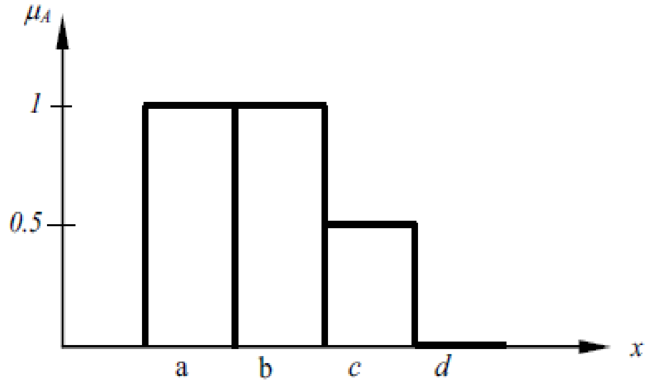
2.1. Basic concepts in the theory of fuzzy sets

The type of data used in estimating the parameters and reliability of probability distribution is of great importance in the accuracy of the results that we will obtain. Therefore, the data type must be specified. One of these types of data is the fuzzy data which is one of the new and important trends of statistics because many phenomena in our real world do not have definite boundaries. They as well do not have accuracy in its measurements. So, we will touch on some basic concepts and important in the theory of fuzzy sets [14].

2.2. Fuzzy sets

For ambiguous set, each element in the fuzzy set has a certain degree of belonging. The fuzzy set was characterized by a membership function that assigns to each element of the set with a degree of membership in the interval [0,1]. The element or object is allowed to belong to a partial membership [15].
Let X be universe of discourse, then the fuzzy subset $\tilde{A}$ characterized by membership function $\mu_{\tilde{A}}(x)$ which produces values between $[0, 1]$ for all x values in the fuzzy sample space X:

$$\tilde{A} = \{(x_i, \mu_{\tilde{A}}(x_i)), x \in \mathrm{X}, i = 1,2,3, \ldots\ldots n, o < \mu_{\tilde{A}}(x) < 1\} \tag{8}$$



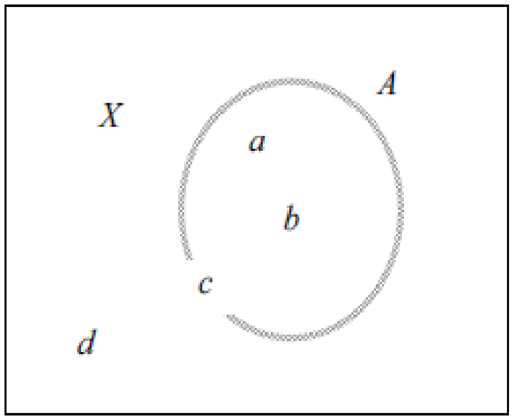


Figure 1. Graphical representation for fuzzy set

2.3. Membership function

The concept of the membership function is the most important in the theory of fuzzy sets. It is used to represent different types of fuzzy sets. A function produces values within the interval [0,1] to reflect the degree of membership of each element in the total set to the fuzzy set. In other word, it is the map that determines the degree of correctness (degree of verification of membership) for the belonging of each element in the overall set to the fuzzy set. It is a function with a non-negative value and the basic condition for this function is to have a range between zero and one [16, 17].

Fuzzy numbers are used to describe uncertainty which are often triangular, trapezoidal, or any other forms [18, 19]. The fuzzy number is a fuzzy set under the following conditions:

1. Convex and normalized.
2. The μ_a belonging function is semi-continuous from the top.
3. The α-specific level group is for each α ε [0,1].
4. Defined on the set of real numbers R

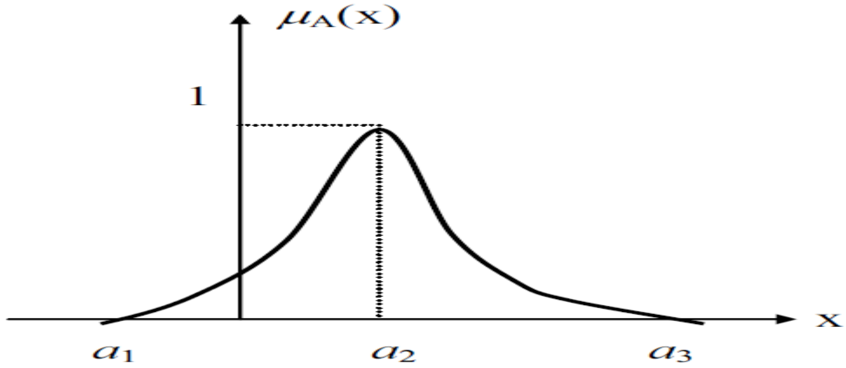


Figure 2. The fuzzy number

2.4. Triangular fuzzy number

It is known as a_1, a_2, a_3, $a_1 < a_2 < a_3$. It is triangle within the interval$[a_1, a_3]$. Its head is at $x = a_2$, and it can be written as $\widetilde{N} = (a_1/a_2/a_3)$ and its membership function is:

$$\mu_{\widetilde{N}}(x) = \begin{cases} \frac{x-\boldsymbol{a_1}}{\boldsymbol{a_2}-\boldsymbol{a_1}} & \boldsymbol{a_1} \le x \le \boldsymbol{a_2} \\ \frac{\boldsymbol{a_3}-x}{\boldsymbol{a_3}-\boldsymbol{a_2}} & \boldsymbol{a_2} \le x \le \boldsymbol{a_3} \\ 0 & o.w \end{cases} \tag{9}$$

2.5. Trapezoidal fuzzy number

It is known as $a_1, a_2, a_3, a_3, , a_1 < a_2 < a_3 < a_4$, and triangle the interval $[a_1, a_4]$. Its head is at $[a_2, a_3]$, and it can be written as $\widetilde{M} = (a_1/a_2, a_3/a_4)$, and its membership function is [20]:

$$\mu_{\widetilde{M}}(x) = \begin{cases} \frac{x-\boldsymbol{a_1}}{\boldsymbol{a_2}-\boldsymbol{a_1}} & \boldsymbol{a_1} \le x \le \boldsymbol{a_2} \\ 1 & \boldsymbol{a_2} \le x \le \boldsymbol{a_3} \\ \frac{\boldsymbol{a_4}-x}{\boldsymbol{a_4}-\boldsymbol{a_3}} & \boldsymbol{a_3} \le x \le \boldsymbol{a_4} \\ 0 & o.w \end{cases} \tag{10}$$

2.6. Fuzzy sample space

It is fuzzy parts $\tilde{x} = (\tilde{x}_1, \dots \dots \dots, \tilde{x}_n)$ from $= (X_1, \dots \dots \dots, X_n)$. Fuzzy sets for X with membership functions has Borel Measure, and satisfy the orthogonally constraint [8, 17]:

$$\sum_{\tilde{x} \in X} \mu_{\tilde{x}}(x) = 1 \tag{11}$$

Also, it is termed as fuzzy information system (FIS).

2.7. Fuzzy event

If $X = (X_1, \dots \dots \dots, X_n)$ in space and B_x is smallest Borel field in X. Then, the fuzzy event is fuzzy subset $\widetilde{A}$ in which its membership has measurable Borel field [17].

2.8. Fuzzy reliability

Reliability was defined as the probability that the unit or device will remain valid after a period of time (t) on use. If T is a continuous random variable, T> 0, the reliability function is [7, 18, 20]:

$$R_T(t) = P(T \geq t) = \int_t^{\infty} f_T(x)dx = 1 - \int_0^t f_T(x)dx = 1 - F_T(t) \tag{12}$$

Its properties are:

- $R(0) = p(T < 0) = 1$
- $R(\infty) = 1$
- $0 \leq R(t) \leq 1$
- If $t_1 < t_2$then, $R(t_1) \geq R(t_2)$

Now, we can say that the fuzzy reliability represents the probability of the unit performing the work required. It is with varying degrees of success for a specified period of time under normal conditions and symbolized by $\widetilde{R}$, which is a function in the fuzzy set $\widetilde{A}$:

$\tilde{R} = \mu_{\tilde{A}_i}(R).R$, While $R(t) = \int_t^{\infty} f(t)dt$ *then*, $\tilde{R} = \mu_{\tilde{A}_i}(R).\int_t^{\infty} f(t)dt, R(t) = 1 - \exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)$

We will assume that the values of the random variable $\widetilde{T}$ are fuzzy number, $\tilde{t} = \{[0,\infty), \mu_{\tilde{t}_i}\}$; $\tilde{k}t\ \tilde{t} =$ و $t \in T$

So, the fuzziness is a real triangular fuzzy number, and:

$$\mu_{\tilde{k}}(k) = \begin{cases} \frac{k-k_{min}}{1-k_{min}} & k \in (k_{min}, 1) \\ \frac{k-k_{max}}{1-k_{max}} & k \in (1, k_{max}) \\ 0 & o.w \end{cases} \tag{13}$$

Where, $0 < k_{min} \leq 1 \leq k_{max}$.

If the random variable T has a traditional fractional distribution with $Frechet(\alpha,\beta)$, the corresponding $\tilde{T}$ of random variable will have $\boldsymbol{\widetilde{Frechet}(\tilde{\alpha},\tilde{\beta})}$ variable.

For each $t \in [0,\infty)$, the Cumulative Fuzzy Distribution Function is:

$$\boldsymbol{\widetilde{F}(\tilde{t})} = \mathbf{exp}\left(-\left(\frac{\hat{\beta}}{\tilde{k}t}\right)^{\hat{\alpha}}\right) \ ;\ \tilde{t} > 0 \tag{14}$$

Then, the fuzzy reliability function is:

$$\boldsymbol{\widetilde{R}(t) = 1 - exp\left(-\left(\frac{\hat{\beta}}{\tilde{k}t}\right)^{\hat{\alpha}}\right)} \tag{15}$$

2.9. Fuzzy maximum likelihood method

Let $x = (x_1, \dots\dots\dots, x_n)$ that is based on Frechet distribution and let $X = (X_1, \dots\dots, X_n)$ that is random vector representing the sample space. The likelihood function for complete data (not fuzzy) is [8, 7, 11, 12, 21]:

$$f(\alpha,\beta;x) = \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} x^{-(\alpha+1)} \exp\left(-\sum_{i=1}^{n}\left(\frac{\beta}{x}\right)^{\alpha}\right) \tag{16}$$

Where X is clearly visible and available with full information about crisp vector. Now, consider the problem that x is not clearly and accurately seen and partially available in a fuzzy subset form with a $\mu_{\tilde{A}}(x)$ having a Borel measurement. The fuzzy set x can express the partial belonging of X from the X random vector. The function $\mu_{\tilde{A}}$ is a probability distribution that explains the limitations of that partial observation $\tilde{x}$. The fuzzy set x can be described by two steps:

- x is drawn from X.
- The vector x viewer after such partial information is encoded in $\mu_{\tilde{A}}(x)$.

The information on x can be represented by the following probability distribution:

$$\mu_{\tilde{x}}(x) = \mu_{\tilde{x}_1}(x)x \dots\dots\dots.. x\mu_{\tilde{x}_n}(x) \tag{17}$$

If x is given and its function is assumed to have a Borel measurement. We can calculate its probability according to the definition of fuzzy probability. The maximum likelihood function of fuzzy data can be obtained as follows:

$$L^* = log\left(L_0(\alpha,\beta;\tilde{x})\right) = nlog(\alpha) + n\alpha\, log(\beta) + \sum_{i=1}^{n} log\left(\int_0^{\infty} x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx\right) \tag{18}$$

Estimates of the ML of α and β can be obtained by maximizing L* and partial derivation for α and β with equivalence of of zero as follows:

$$=\frac{n}{\hat{\alpha}} + nlog(\hat{\beta}) - \left[\sum_{i=1}^{n} \frac{\int_0^\infty \left[\left(x^{-(\hat{\alpha}+1)}.ln(x)+x^{-2\hat{\alpha}-1}(\hat{\beta})^{\hat{\alpha}}.ln\left(\frac{\hat{\beta}}{x}\right)\right)exp\left(-\left(\frac{\hat{\beta}}{x}\right)^{\hat{\alpha}}\right).\mu_{\tilde{x}_i}(x)dx.\right]}{\int_0^\infty x^{-(\hat{\alpha}+1)}\, exp\left(-\left(\frac{\hat{\beta}}{x}\right)^{\hat{\alpha}}\right)\mu_{\tilde{x}_i}(x)dx}\right] = 0 \quad (19)$$

$$\frac{\partial L^*}{\partial \beta} = \frac{n\hat{\alpha}}{\hat{\beta}} - \sum_{i=1}^{n} \frac{\int_0^\infty x^{-\alpha-1}\left(\frac{\beta}{x}\right)^{\hat{\alpha}} \frac{\hat{\alpha}}{\hat{\beta}} exp(-\left(\frac{\beta}{x}\right)^{\hat{\alpha}})\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{-(\hat{\alpha}+1)}\, exp\left(-\left(\frac{\hat{\beta}}{x}\right)^{\hat{\alpha}}\right)\mu_{\tilde{x}_i}(x)dx} = 0 \quad (20)$$

We can see from the formulas (19) and (20) that there is no closed formula for the solution. So, we will look for frequently numerical methods to obtain the ML via Newton-Raphson method to obtain the estimates of the $\hat{\alpha}_{fmle}$ and $\hat{\beta}_{fmle}$ as following:

Let $\theta = (\alpha, \beta)^T$ represent the parameter vector, then at step $(h+1)$ from iterative steps, we obtain on parameters as following*:*

$$\theta^{h+1} = \theta^h - \left[\frac{\partial^2 L^*(\alpha,\beta;\tilde{x})}{\partial\theta\partial\theta^T}\Big|_{\theta=\theta^h}\right]^{-1} . \left[\frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\theta}\Big|_{\theta=\theta^h}\right] \quad (21)$$

$$\frac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\theta} = \begin{pmatrix} \dfrac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha} \\ \dfrac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\beta} \end{pmatrix}$$

$$\frac{\partial^2 L^*(\alpha,\beta;\tilde{x})}{\partial\theta\partial\theta^T} = \begin{pmatrix} \dfrac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha^2} & \dfrac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha\partial\beta} \\ \dfrac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\alpha\partial\beta} & \dfrac{\partial L^*(\alpha,\beta;\tilde{x})}{\partial\beta^2} \end{pmatrix}$$

$$\frac{\partial L^{*2}}{\partial\alpha^2} = \frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{1}{\left(\int_0^\infty x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)}\left(\int_o^\infty \ln(x)^2\left(\frac{\beta}{x}\right)^\alpha + 2\ln(x)\ln\left(\frac{\beta}{x}\right) - ln\left(\frac{\beta}{x}\right)^2 + \right.\right.$$
$$\left(\frac{\beta}{x}\right)^\alpha ln\left(\frac{\beta}{x}\right)^2\right).\left(x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right) + \frac{\left(\int_0^\infty \ln(x)+\left(\frac{\beta}{x}\right)^\alpha ln\left(\frac{\beta}{x}\right)\right).\left(x^{-\alpha-1} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}{\left(\int_0^\infty x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}$$

(22)

$$\frac{\partial L^{*2}}{\partial\beta^2} = -\frac{n\alpha}{\beta^2} - \sum_{i=1}^{n}\frac{\left(\frac{\alpha^2}{\beta^2}-\frac{\alpha}{\beta^2}-\frac{\alpha^2}{\beta^2}\left(\frac{\beta}{x}\right)^\alpha\right).\left(x^{-\alpha-1} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right)\mu_{\tilde{x}_i}(x)dx}{\int_0^\infty x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx} + \frac{\int_0^\infty x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha\frac{\alpha}{\beta}exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}) \quad (23)$$

$$\frac{\partial L^{*2}}{\partial\alpha\partial\beta} = -\frac{n}{\beta} - \sum_{i=1}^{n}\left(\frac{-\ln(x)\frac{\alpha}{\beta}+\frac{\alpha}{\beta}+\frac{1}{\beta}-\left(\frac{\beta}{x}\right)^\alpha\ln(x)\frac{\alpha}{\beta}).\left(x^{-\alpha-1} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\right)\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)} + \right.$$
$$\left.\frac{\int_0^\infty x^{-\alpha-1}\ln(x)\, exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)+x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha ln\left(\frac{\beta}{x}\right)exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx.\int_0^\infty x^{-\alpha-1}\left(\frac{\beta}{x}\right)^\alpha\frac{\alpha}{\beta}exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx}{\left(\int_0^\infty x^{-(\alpha+1)} exp\left(-\left(\frac{\beta}{x}\right)^\alpha\right)\mu_{\tilde{x}_i}(x)dx\right)^2}\right) \quad (24)$$

We continue with replication until $\left|\left|\theta^{h+1} - \theta^h\right|\right|$ is less than ε , wher $\varepsilon > 0$, a very small positive number.

2.10. Fuzzy Bays method

Bayes theorem assumes that the unknown parameters are random variables and that prior information is given in the form of a probability distribution known as the prior probability function. This information is identified from previous data and experiments of the theory that governs the phenomenon. The Bayes theory also depends on the current information of the sample. It can represent the likelihoods function of the observations by combining the probability density function of the parameters with the maximum likelihood function of the current observations. Accordingly, we can obtain the posterior probability distribution and under a quadratic loss function, we derive the Bayes estimates [4, 5, 9, 17].

Let $x = (x_1, \dots\dots\dots, x_n)$ that is based on Frechet distribution and let $X = (X_1, \dots\dots, X_n)$ that is random vector represent the sample space. The likelihood function for complete non-fuzzy data is:

$$f(\alpha,\beta;x) = \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} x^{-(\alpha+1)} \exp\left(-\sum_{i=1}^{n}\left(\frac{\beta}{x}\right)^{\alpha}\right) \quad (25)$$

Where X is clearly visible and available with full information about crisp vector.
Now consider the problem that x is not clearly and accurately seen and partially available in a fuzzy subset form with a $\mu_{\tilde{A}}(x)$ have a Borel measurement. Let $\pi_1(\alpha)$ and $\pi_2(\beta)$ represent the priors distribution. We assumed that as Gamma distribution as following:

$$\pi_1(\alpha) = \frac{d^c}{\Gamma c}\alpha^{c-1}\exp(-\alpha d) \quad (26)$$

$$\pi_2(\beta) = \frac{b^a}{\Gamma a}\beta^{a-1}\exp(-\beta b) \quad (27)$$

Where $\alpha \sim \text{gamma}(c,d), \beta \sim \text{gamma}(a,b)$.

$$\pi_1(\alpha,\beta \mid \tilde{x}) = \frac{\pi_1(\alpha).\pi_2(\beta).\ell(\alpha,\beta;\tilde{x})}{\iint \pi_1(\alpha).\pi_2(\beta).\ell(\alpha,\beta;\tilde{x})d\alpha d\beta} \quad (28)$$

$$\ell(\alpha,\beta;\tilde{x}) = \alpha^{(n)}\beta^{(\alpha n)}\exp(-\alpha d)\exp(-\beta b)\prod_{i=1}^{n}\int x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx \quad (29)$$

$$\ell(\alpha,\beta;\tilde{x}) = \alpha^n\beta^{n\alpha}\prod_{i=1}^{n}\int_0^{\infty} x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}dx \quad (30)$$

The posterior is:

$$f(\theta / x) = \frac{k(\theta,x)}{\int k(\theta,x)} \quad (31)$$

$$f(\alpha,\beta / \tilde{x}) = \frac{\alpha^{(n+c-1)}\beta^{(\alpha n+a-1)}\exp(-\alpha d)\exp(-\beta b)\prod_{i=1}^{n}\int x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx}{\int\int \alpha^{(n+c-1)}\beta^{(\alpha n+a-1)}\exp(-\alpha d)\exp(-\beta b)\prod_{i=1}^{n}\int x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx d\beta d\alpha} \quad (32)$$

Under loss function $g(\alpha,\beta)$, we get:

$$E(g(\alpha,\beta \mid \tilde{x})) = \frac{\int\int g(\alpha,\beta)\alpha^{(n+c-1)}\beta^{(\alpha n+a-1)}\exp(-\alpha d)\exp(-\beta b)\prod_{i=1}^{n}\int x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx d\beta d\alpha}{\int\int \alpha^{(n+c-1)}\beta^{(\alpha n+a-1)}\exp(-\alpha d)\exp(-\beta b)\prod_{i=1}^{n}\int x^{-(\alpha+1)}\exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\mu_{\tilde{x}_i}(x)dx d\beta d\alpha} \quad E(g(\alpha,\beta \mid \tilde{x})) = \frac{\int\int g(\alpha,\beta)e^{nQ}d\beta d\alpha}{\int\int e^{nQ}d\beta d\alpha} \quad (33)$$

$$Q(\alpha,\beta) = ln(\pi_1(\alpha)\pi_2(\beta)\ell(\alpha,\beta;\tilde{x}) \equiv \rho(\alpha;\beta) + \mathcal{L}(\alpha;\beta) \quad (34)$$

$$E(g(\alpha,\beta \mid \tilde{x})) = \frac{\int\int e^{n\bar{F}}d\beta d\alpha}{\int\int e^{nF}d\beta d\alpha} \quad (35)$$

We will use Tierney and Kandane's approximation algorithm to obtain (α)ˆ_bays and βˆ_bays by assuming initial values for $(\acute{\alpha},\alpha,\beta)$ as following:

$$\hat{g}(\alpha,\beta) = \left(\frac{\bar{H}}{H}\right)^{1/2} exp\left(n\left(\bar{F}(\acute{\alpha},\acute{\beta}) - F(\alpha,\beta)\right)\right) \quad (36)$$

$$\bar{F}(\acute{\alpha},\acute{\beta}) = \frac{\left(ln(\acute{\alpha})+ln\left(\acute{\alpha}^{n+c-1}\beta^{n\alpha+a-1}exp(-\acute{\alpha}d)\,exp(-\beta b)\prod\int x^{-(\acute{\alpha}+1)}exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right)\mu_{\tilde{x}_i}(x)dx\right)}{n} \quad (37)$$

$$F(\acute{\alpha},\beta) = \frac{\left(ln\left(\alpha^{n+c-1}\beta^{n\alpha+a-1}exp(-\alpha d)\,exp(-\beta b)\prod\int x^{-(\alpha+1)}exp\left(-\left(\frac{\beta}{x}\right)^{\alpha}\right)\right)\mu_{\tilde{x}_i}(x)dx\right)}{n} \quad (38)$$

2.11. Simulation experiments

The Monte-Carlo simulation method was used to compare the estimation methods of the reliability function for Frechet distribution and to illustrate the effect of the estimation method of the reliability function for the following:

1. Change in sample size.
2. Change in the relationship between the α shape parameter and the parameter β.
3. Choosing sample sizes (10,50,100,150,500)
4. Choosing Experimental values for α and β as following:

Table 1. The values of α and β

Parameter	α	β	α	β	α	β
Value	0.50	0.50	0.75	0.75	2.0	2.0
	0.50	1.00	2.00	3.00		

2.10.1 Data generating

The generation of a variable follows a uniform distribution u ~ U (0,1) using the rand term. Generate fuzzy data following the Frechet distribution by inverse transformation method using the following formula:

$$t_i = \beta \left(Ln\left(\frac{1}{u}\right)\right)^{-\frac{1}{\alpha}} \quad ; \; i = 1,2, \ldots \ldots, n \qquad (39)$$

The sample is represented by vector t of Frechet distribution. The t-sample vector is converted to fuzzy using the fuzzy hypothetical information system as in Figure. 3 corresponding to the following membership functions:

$$\mu_{\tilde{t}_1}(t) = \begin{cases} 1 & t \leq 0.05 \\ \frac{0.25-t}{0.2} & 0.05 \leq t \leq 0.25 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_2}(t) = \begin{cases} \frac{t-0.05}{0.2} & 0.05 \leq t \leq 0.25 \\ \frac{0.5-t}{0.25} & 0.25 \leq t \leq 0.5 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_3}(t) = \begin{cases} \frac{t-0.25}{0.25} & 0.25 \leq t \leq 0.5 \\ \frac{0.75-t}{0.25} & 0.5 \leq t \leq 0.75 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_4}(t) = \begin{cases} \frac{t-0.5}{0.25} & 0.5 \leq t \leq 0.75 \\ \frac{1-t}{0.25} & 0.75 \leq t \leq 1 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_5}(t) = \begin{cases} \frac{t-0.75}{0.25} & 0.75 \leq t \leq 1 \\ \frac{1.5-t}{0.5} & 1 \leq t \leq 1.5 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_6}(t) = \begin{cases} \frac{t-1}{0.5} & 1 \leq t \leq 1.5 \\ \frac{2-t}{0.5} & 1.5 \leq t \leq 2 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_7}(t) = \begin{cases} \frac{t-1.5}{0.5} & 1 \leq t \leq 1.5 \\ 3-t & 2 \leq t \leq 3 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{t}_8}(t) = \begin{cases} t-2 & 2 \leq t \leq 3 \\ 1 & t \geq 3 \\ 0 & o.w \end{cases}$$

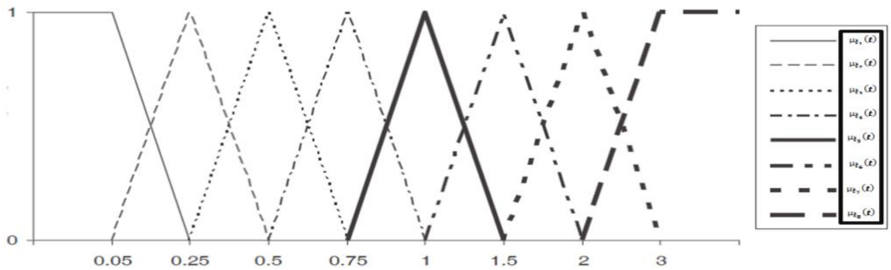


Figure 3. The Fuzzy Hypothetical information system used in simulating simulation data

The estimates of α and β for the fuzzy sample were obtained using NR method in the maximum likelihood method. In Bayes method, we used a Tierney and Kandane's approximation algorithm to obtain Bays estimates, and to calculate Bays estimates. We assume that α and β have an initial gamma distribution (α ~ gamma (a, b), β ~ gamma (c, d). We suggest that the prior distributions of the parameters are incomplete. Press in 2011, suggests using very small positive values for the meta parameters in the initial distribution. Hence, we will assume a = b = c = d = 0.0001 and the assumed sample size and iterations are 1000 times for each simulation experiment to obtain homogenous in estimating the reliability function of the Frechet distribution. The simulation experiments were carried out using the MATLAB programming language.
The values of the reliability function were generated using the following formula:

$$\tilde{R}(t_i) = 1 - exp\left(-\left(\frac{\hat{\beta}}{\tilde{t}_i}\right)^{\hat{\alpha}}\right) ; i = 1,2, \dots \dots, n \qquad (40)$$

After the creation of the randomized fuzzy values $\tilde{t}_i$ of the CDF function according to the size of the given samples and the default values of initial parameters according to the formula ($\tilde{R}(t_i)$), the values of t_i and the initial parameters were computed according to the functions of the $\mu_{\tilde{t}_i}(t)$ for each fuzzy unit $\tilde{t}_i$ .Then, extract for each $\tilde{R}(t_i)$ and find expectation of $\tilde{R}(t_i)$ as follows:

$$\tilde{R}(t) = \hat{E}(\tilde{R}(t_i)/\tilde{x}_i) = \frac{1}{K}\sum_{h=1}^{K} R^{(h)}(t) \qquad (41)$$

Comparison of simulation results: Simulation results are compared using the following statistical measures:

$$\mathrm{MSE}(\tilde{R}) = \frac{1}{K}\sum_{i=1}^{K}(\tilde{R} - R)^2 \qquad (42)$$

$$MAPE(\tilde{R}) = \frac{1}{K}\sum_{i=1}^{k}\left|\frac{\tilde{R}-R}{R}\right| \qquad (43)$$

Table 2. MSE and MAPE for parameters and reliability for $\alpha = 0.5$, $\beta = 0.5$

n		**MLE**			**Bays**		
		Estimate	***MSE***	***MAPE***	***Estimate***	***MSE***	***MAPE***
10	$\hat{\alpha}$	0.665352	0.004675	0.330703	**0.614329**	0.001538	0.228658
	$\hat{\beta}$	0.351727	0.004442	0.297344	**0.585701**	0.000934	0.171403
	$\tilde{R}$	0.602032	0.026839	1.395128	**0.746900**	**0.001419**	**0.491027**
50	$\hat{\alpha}$	0.717622	0.001013	0.087049	**0.524356**	0.000014	0.009742
	$\hat{\beta}$	0.413690	0.000185	0.034524	**0.525714**	0.000016	0.010286
	$\tilde{R}$	0.660551	0.001221	0.449476	**0.715346**	**0.000099**	**0.132370**
100	$\hat{\alpha}$	0.687524	0.000386	0.037505	**0.509676**	0.000001	0.001935
	$\hat{\beta}$	0.401543	0.000126	0.019691	**0.510069**	0.000001	0.002014
	$\tilde{R}$	0.674039	0.001568	0.459365	**0.698277**	**0.000015**	**0.051181**
150	$\hat{\alpha}$	0.696143	0.000264	0.026152	**0.506274**	0.000001	0.000837
	$\hat{\beta}$	0.417060	0.000047	0.011059	**0.504268**	0.000001	0.000569
	$\tilde{R}$	0.675177	0.000671	0.363850	**0.702000**	**0.000004**	**0.024439**
500	$\hat{\alpha}$	0.693670	0.000076	0.007747	**0.502247**	0.000001	0.000090
	$\hat{\beta}$	0.407790	0.000018	0.003688	**0.502485**	0.000001	0.000099
	$\tilde{R}$	0.675183	0.000887	0.408905	**0.701440**	**0.000001**	**0.012886**

Table 3. MSE and MAPE for parameters and reliability for $\alpha = 0.75$, $\beta = 0.75$

n		**MLE**			**Bays**		
		Estimate	***MSE***	***MAPE***	***Estimate***	***MSE***	***MAPE***
10	$\hat{\alpha}$	1.031283	0.016770	0.506585	**0.864063**	0.001781	0.152084
	$\hat{\beta}$	0.632140	0.002330	0.169113	**0.839404**	0.001100	0.119206
	$\tilde{R}$	0.669902	0.003624	0.583852	**0.718141**	**0.001088**	**0.445143**
50	$\hat{\alpha}$	1.054480	0.002097	0.081195	**0.776431**	0.000016	0.007048
	$\hat{\beta}$	0.723381	0.000028	0.008296	**0.774335**	0.000013	0.006489
	$\tilde{R}$	0.675416	0.000082	0.126202	**0.693642**	**0.000076**	**0.119272**

n		MLE			Bays		
		Estimate	***MSE***	***MAPE***	***Estimate***	***MSE***	***MAPE***
100	$\hat{\alpha}$	1.013391	0.000779	0.035119	**0.761222**	0.000002	0.001496
	$\hat{\beta}$	0.710599	0.000021	0.005253	**0.761430**	0.000002	0.001524
	$\tilde{R}$	0.695482	0.000096	0.124914	**0.704118**	**0.000019**	**0.056473**
150	$\hat{\alpha}$	1.034828	0.000560	0.025318	**0.758009**	0.000001	0.000712
	$\hat{\beta}$	0.712536	0.000011	0.003330	**0.759399**	0.000001	0.000835
	$\tilde{R}$	0.692972	0.000055	0.103427	**0.712085**	**0.000012**	**0.045662**
500	$\hat{\alpha}$	1.017954	0.000147	0.007145	**0.752146**	0.000001	0.000057
	$\hat{\beta}$	0.720460	0.000003	0.000788	**0.752664**	0.000001	0.000071
	$\tilde{R}$	0.701109	0.000059	0.105242	**0.704801**	**0.000001**	**0.012883**

Table 4. MSE and MAPE for parameters and reliability for $\alpha = 2, \beta = 2$

n		MLE			Bays		
		Estimate	***MSE***	***MAPE***	***Estimate***	***MSE***	***MAPE***
10	$\hat{\alpha}$	0.961496	0.111377	0.519252	**2.095691**	0.001278	0.047845
	$\hat{\beta}$	1.978953	0.000049	0.010524	**2.097172**	0.001099	0.048586
	$\tilde{R}$	0.656074	0.001992	0.430734	**0.750037**	**0.001255**	**0.460582**
50	$\hat{\alpha}$	0.761001	0.030734	0.123900	**2.013586**	0.000006	0.001359
	$\hat{\beta}$	1.983040	0.000006	0.001696	**2.014768**	0.000008	0.001477
	$\tilde{R}$	0.667898	0.002216	0.627775	**0.715095**	**0.000037**	**0.065543**
100	$\hat{\alpha}$	0.752470	0.015581	0.062377	**2.010086**	0.000001	0.000504
	$\hat{\beta}$	1.984084	0.000003	0.000796	**2.009738**	0.000001	0.000487
	$\tilde{R}$	0.664564	0.001710	0.567223	**0.705900**	**0.000013**	**0.044867**
150	$\hat{\alpha}$	0.760166	0.010257	0.041328	**2.006478**	0.000001	0.000216
	$\hat{\beta}$	1.983691	0.000002	0.000544	**2.005509**	0.000001	0.000184
	$\tilde{R}$	0.662468	0.001601	0.562565	**0.703427**	**0.000004**	**0.025312**
500	$\hat{\alpha}$	0.760789	0.003072	0.012392	**2.002381**	0.000001	0.000024
	$\hat{\beta}$	1.985192	0.000001	0.000148	**2.002111**	0.000001	0.000021
	$\tilde{R}$	0.660134	0.001261	0.506470	**0.700601**	**0.000001**	**0.009800**

Table 5. MSE and MAPE for parameters and reliability for $\alpha = 0.5, \beta = 1$

n		MLE			Bays		
		Estimate	***MSE***	***MAPE***	***Estimate***	***MSE***	***MAPE***
10	$\hat{\alpha}$	**0.480387**	0.000066	0.039764	0.581383	0.000875	0.162766
	$\hat{\beta}$	**0.948387**	0.000290	0.051613	1.124961	0.001814	0.124961
	$\tilde{R}$	**0.680231**	**0.000084**	**0.124583**	0.717208	0.000569	0.320202
50	$\hat{\alpha}$	0.497147	0.000001	0.001169	**0.522001**	0.000012	0.008800
	$\hat{\beta}$	0.948844	0.000058	0.010231	**1.017313**	0.000008	0.003463
	$\tilde{R}$	0.696751	0.000075	0.115406	**0.709575**	**0.000019**	**0.055026**
100	$\hat{\alpha}$	0.497423	0.000001	0.000526	**0.511398**	0.000002	0.002280
	$\hat{\beta}$	0.953799	0.000023	0.004620	**1.007785**	0.000001	0.000778
	$\tilde{R}$	0.690411	0.000060	0.105657	**0.703406**	**0.000006**	**0.028394**
150	$\hat{\alpha}$	0.497686	0.000001	0.000309	**0.504592**	0.000001	0.000459
	$\hat{\beta}$	0.954672	0.000014	0.003022	**1.005559**	0.000001	0.000278
	$\tilde{R}$	0.691585	0.000055	0.103434	**0.700917**	**0.000001**	**0.016393**
500	$\hat{\alpha}$	0.498290	0.000001	0.000068	**0.502093**	0.000001	0.000084
	$\hat{\beta}$	0.953149	0.000004	0.000937	**1.001977**	0.000001	0.000040
	$\tilde{R}$	0.694683	0.000055	0.105597	**0.702256**	**0.000001**	**0.006456**

Table 6. MSE and MAPE for parameters and reliability for $\alpha = 2, \beta = 3$

n		MLE			Bays		
		Estimate	***MSE***	***MAPE***	***Estimate***	***MSE***	***MAPE***
10	$\hat{\alpha}$	**0.775972**	0.000527	0.040430	2.106751	0.001504	0.053376
	$\hat{\beta}$	**1.987516**	0.000025	0.007327	3.072710	0.000700	0.024237
	$\tilde{R}$	**0.703430**	**0.000017**	**0.036894**	0.735256	0.000358	0.234907
50	$\hat{\alpha}$	1.178034	0.014653	0.082197	**2.016922**	0.000008	0.001692
	$\hat{\beta}$	2.991681	0.000001	0.000555	**3.020808**	0.000011	0.001387
	$\tilde{R}$	0.675872	0.000947	0.348272	**0.702239**	**0.000024**	**0.062531**
100	$\hat{\alpha}$	1.158120	0.008573	0.042094	**2.009988**	0.000001	0.000499
	$\hat{\beta}$	2.992516	0.000001	0.000249	**3.011038**	0.000001	0.000368
	$\tilde{R}$	0.677362	0.000883	0.351342	**0.700187**	**0.000007**	**0.034524**
150	$\hat{\alpha}$	1.029419	0.006931	0.032353	**2.007362**	0.000001	0.000245
	$\hat{\beta}$	2.991553	0.000000	0.000188	**3.004735**	0.000001	0.000105
	$\tilde{R}$	0.670388	0.001062	0.404797	**0.708897**	**0.000002**	**0.015307**
500	$\hat{\alpha}$	1.746605	0.008385	0.014207	**2.002305**	0.000001	0.000023
	$\hat{\beta}$	2.993042	0.000001	0.000050	**3.002757**	0.000001	0.000018
	$\tilde{R}$	0.680761	0.000644	0.343294	**0.697266**	**0.000001**	**0.008551**

3. Applied side

Cancer is a serious and rapidly spreading disease. Every year millions of people are infected with this disease. This deadly disease is the main factor in research and new discoveries made by scientists in the field of treatment. New technology devices developed as a result of these efforts offer new sources of cancer diagnosis and treatment. Therefore, the importance of the devices that reveal the disease was to be addressed. One of the most important is the linear accelerator device, which is a modern and advanced device in the detection of cancer and radiation treatment.

3.1. Linear accelerator

Linear accelerator device is one of the advanced and modern devices in the detection and destruction of cancer cells by radiation. This device is used in the following cases:

• To treat cancer by destroying cancer cells.
• Control cancer by preventing cancer cells from growing and spreading.
• Relieving cancer symptoms such as pain.

The device is one of the advanced medical equipment used to treat tumors as in the center of Babylon for the treatment of tumors. For a linear accelerator providing service to citizens on a continuous basis, the center operates only one device for the service. But, in the event of malfunction of the device or stopped working for technical reasons, they operate the second device to provide continuous service. However, it is noted that there is no accurate recording of the times of operation of the device and the times of stopping the device. The operators of the device must know the operating periods and holidays accurately in the case of malfunction of the device. For example, the device operator must inform the management of the center orally and the hospital management. In turn, the management must call the company that equipped the device to carry out the repair of device imprecision in recording operating times and holidays. This leads us to say that data on the time of operation of the linear accelerator are fuzzy numbers and belong to the fuzzy times with certain degrees of membership.

Approximate information was obtained on the length of operation of the equipment until the work of the specialists in charge of the device, the supervising engineers and the administration of the center. These times were arranged in Table 7, measured in months for the period from the beginning of installation of the equipment at the center:

Table 7. Extend the speed of the linear accelerator system until it stops working in months

i	t_i	i	t_i	i	t_i	i	t_i	i	t_i	i	t_i	i	t_i
1	1.3	11	2.0	21	2.6	31	3.5	41	4.6	51	6.5	61	12.6
2	1.4	12	2.0	22	2.8	32	3.6	42	5.1	52	6.6	62	14.0
3	1.4	13	2.0	23	3.0	33	3.7	43	5.2	53	6.8	63	17.7

i	t_i	i	t_i	i	t_i	i	t_i	i	t_i	i	t_i	i	t_i
4	1.6	14	2.1	24	3.0	34	3.8	44	5.3	54	7.7		
5	1.7	15	2.2	25	3.1	35	3.8	45	5.5	55	8.5		
6	1.7	16	2.2	26	3.3	36	3.9	46	5.8	56	10.4		
7	1.8	17	2.4	27	3.3	37	4.1	47	5.8	57	10.6		
8	1.8	18	2.4	28	3.3	38	4.2	48	5.9	58	10.6		
9	1.9	19	2.4	29	3.4	39	4.5	49	6.1	59	10.9		
10	2.0	20	2.5	30	3.5	40	4.5	50	6.3	60	11.4		

3.2. Data fuzzification

The real sample vector x was converted to mist using the fuzzy information system as in Figure. 4 corresponding to the following functions:

$$\mu_{\tilde{t}_1}(x) = \begin{cases} 1 & x \le 1.3 \\ \frac{1.7-x}{0.4} & 1.3 \le x \le 1.7 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-1.3}{0.4} & 1.3 \le x \le 1.7 \\ \frac{1.8-x}{0.1} & 1.7 \le x \le 1.8 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-1.7}{0.1} & 1.7 \le x \le 1.8 \\ \frac{2.1-x}{0.3} & 1.8 \le x \le 2.1 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-1.8}{0.3} & 1.8 \le x \le 2.1 \\ \frac{2.6-x}{0.5} & 2.1 \le x \le 2.6 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \frac{x-2.1}{0.3} & 2.1 \le x \le 2.6 \\ \frac{3.3-x}{0.7} & 2.6 \le x \le 3.3 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_6}(x) = \begin{cases} \frac{x-2.6}{0.7} & 2.6 \le x \le 3.3 \\ \frac{3.8-x}{0.5} & 3.3 \le x \le 3.8 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \frac{x-3.3}{0.5} & 3.3 \le x \le 3.8 \\ \frac{3.9-x}{0.1} & 3.8 \le x \le 3.9 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_8}(x) = \begin{cases} \frac{x-3.8}{0.1} & 3.8 \le x \le 3.9 \\ \frac{5.1-x}{1.1} & 3.9 \le x \le 5.1 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_9}(x) = \begin{cases} \frac{x-3.9}{1.1} & 3.9 \le x \le 5.1 \\ \frac{6.1-x}{1} & 5.1 \le x \le 6.1 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_{10}}(x) = \begin{cases} \frac{x-5.1}{1} & 5.1 \le x \le 6.1 \\ \frac{6.1-x}{4.5} & 6.1 \le x \le 10.6 \\ 0 & o.w \end{cases}$$

$$\mu_{\tilde{x}_{11}}(x) = \begin{cases} \frac{x-6.1}{7.1} & 10.6 \le x \le 17.7 \\ 1 & x \ge 17.7 \\ 0 & o.w \end{cases}$$

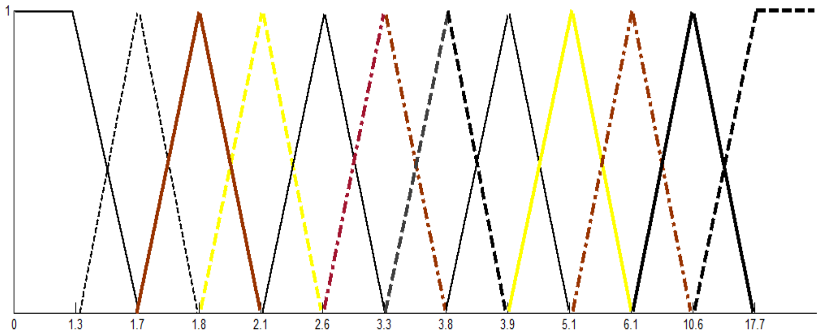


Figure 4. Fuzzy information system used to process real data

3.3. Data analysis

We have used the relationship between the arithmetic mean and the variance for the distribution of the parameter matrix. This is to obtain the initial value of the algorithms used in the estimation of the parameters by solving the nonlinear equations below and using the Newton-Raphson method.

$$M = \beta\Gamma\left(1 - \frac{1}{\alpha}\right) \tag{44}$$

$$S = \beta^2\Gamma\left(1 - \frac{2}{\alpha}\right) - \left(\beta\Gamma\left(1 - \frac{1}{\alpha}\right)\right)^2 \tag{45}$$

The GOF tests mentioned have used to determine the appropriateness of the distribution to represent the duration of the linear accelerator operation. Table 8 summarizes the estimates of the methods of Frechet distribution parameters, the rate of uncertainty function and the values of goodness of fit tests for real data.

Table 8. Results of analysis of real data

$Meth$	$Parameter$			$Type\ of\ test$			
	α	β	$\tilde{R}$	$HQIC$	$CACI$	BIC	ACI
FMl	3.584435	4.2195	0.645402	821.1003	831.816	821.9623	16.25198
$FBays$	2.9057	3.4618	0.557766	599.1445	609.8602	600.0066	15.61715

We note that the four varieties of fit goodness for Bayes method has achieved the lowest value of the maximum likelihood method. This indicates that the duration data of the linear accelerator is more consistent with the Fuzzy Frechet distribution when estimating the parameters of this distribution in Bayes method.

4. Results

- The Bays method exceeds the estimate of the maximum likelihood method. The fuzzy reliability was estimated using Bays method with the lowest MSE and the least MAPE.
- In the fuzzy Bays method, when sample size is larger, the MSE and the MAPE are reduced to as little as the sample size is 500.

- The parameter estimated by Bays method and maximum likelihood method with the default values as well as the fuzzy reliability converges from the default reliability as the size of the sample increases.
- Bayes method has achieved the lowest value of the maximum Likelihood method. This indicates that the duration data of linear accelerator is more consistent with the Fuzzy Frechet distribution when estimating the parameters of this distribution in Bayes.

5. Conclusions

The best method to estimate the fuzzy reliability of the Frichet distribution is by fuzzy Bayes method at large sample sizes.

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