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# Fractional Brownian motion inference of multivariate stochastic differential equations

**Ahmed H. Ali[1], Qutaiba N. Nayef AL-Qazaz[2]**
[1] Department of Statistics, Baghdad University
[2] Head of Statistics Department, Baghdad University

**ABSTRACT**

Recently, the financial mathematics has been emerged to interpret and predict the underlying mechanism that generates an incident of concern. A system of differential equations can reveal a dynamical development of financial mechanism across time. Multivariate wiener process represents the stochastic term in a system of stochastic differential equations (SDE). The standard wiener process follows a Markov chain, and hence it is a martingale (kind of Markov chain), which is a good integrator. Though, the fractional Wiener process does not follow a Markov chain, hence it is not a good integrator. This problem will produce an Arbitrage (non-equilibrium in the market) in the predicted series. It is undesired property that leads to erroneous conclusion, as it is not possible to build a mathematical model, which represents the financial phenomenon. If there is Arbitrage (unbalance) in the market, this can be solved by Wick-Itô-Skorohod stochastic integral (renormalized integral). This paper considers the estimation of a system of fractional stochastic differential equations (FSDE) using maximum likelihood method, although it is time consuming. However, it provides estimates with desirable characteristic with the most important consistency. Langevin method can be used to find the mathematical form of the functions of stochastic differential equations. This includes drift and diffusion by estimating conditional mean and variance from the data and finding the suitable function achieves the least error, and then estimating the parameters of the model by numerical optimal solution search method. Data used in this paper consist of three banking sector stock prices including Baghdad Bank (BBOB), the Commercial Bank (BCOI), and the National Bank (BNOI).

**Keywords**: Fractional Brownian motion, Stochastic differential equations, Maximum likelihood, Hurst index, Langevin method

***Corresponding Author:***

Ahmed Hasan Ali
Department of statistics
Baghdad university
Baghdad, Iraq
E-mail: auroahmed0@gmail.com

## 1. Introduction

The optimal accuracy among all estimators for estimating the Hurst exponent is the maximum likelihood estimator (MLE). Although it is very computationally expensive, their analytical solution is too difficult. So, using numerical search methods is inevitable. Studying multivariate fractional stochastic differential equation can be valuable, because it reveals the dynamic issues that hide the relationship between components. These components act in the same studied phenomenon especially in the finance as group of assets are interact among them [1]. Portfolio can be built upon the volatility or covariance of multivariate assets which enable financial practitioners to select the best group of investment strategy. This paper applies multivariate maximum likelihood to estimate the parameters of system of fractional stochastic differential equations.

### 1.1. Long range dependence

Long range dependence involves $1/f$ type behavior, where $f$ represents the frequency equivalent to $\sum_{i=-\infty}^{\infty} \rho_i = \infty$. This will make conventional methods such as ARIMA are inappropriate due to slow decay of correlation structure. The power law behavior of fractional Brownian motion FBM enables us to model stochastic process with long range dependence like $1/f$ type process. Data with long range dependence and self-similarity exist naturally in real-world models. Studying the dynamics of a phenomenon gives a thorough sight about its behavior and its development.

A fractional Brownian motion FBM is an irregular diffusion process with covariance as shown below [2]:

$$E\left(W_t^H W_s^H\right)=\frac{1}{2}\left\{|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right\} \tag{1}$$

Where $0<H<1$ is Hurst index

and the variance

$$E\left(W_t^H W_t^H\right)=E\left(W_t^H\right)^2=\frac{1}{2}\left\{|t|^{2H}+|t|^{2H}-|t-t|^{2H}\right\}=|t|^{2H} \tag{2}$$

The covariance between two different Wiener process is:

$$E\left(W_{1,t}^{H_1} W_{2,s}^{H_2}\right)=\frac{1}{2}\left\{|t|^{2H_{12}}+|s|^{2H_{12}}-|t-s|^{2H_{12}}\right\} \tag{3}$$

The difference of FBM is called Fractional Gaussian Noise FGN and have variance covariance as:

$$E\left(dW_{i,k_1}^{H_i} dW_{j,k_2}^{H_j}\right)=\frac{1}{2}\left[\left(k_1+1\right)^{2H_{ij}}-\left(k_1+k_2\right)^{2H_{ij}}+\left(k_2-1\right)^{2H_{ij}}\right] \tag{4}$$

So, the likelihood of FGN becomes [3,4] :

$$p(\underline{x},H)=\frac{1}{(2\pi)^{\frac{N}{2}}\left|\Omega_H\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\underline{x}'\Omega_H^{-1}\underline{x}\right\} \tag{5}$$

Where, $\Omega$ can be calculated by [5]:

$$\Omega=E\left(dW_{i,k_1}^{H_i} dW_{j,k_2}^{H_j}\right)=\frac{\sigma^2}{2}\left[\left(k_1+1\right)^{2H_{ij}}-\left(k_1+k_2\right)^{2H_{ij}}+\left(k_2-1\right)^{2H_{ij}}\right] \tag{6}$$

Explicit form for the estimate of Hurst index is impossible to be obtained, as a function of the data. However, the maximum of its object function could be found by numerical methods.

FBm can be represented as a weighted average of standard Brownian motion (i.e., when $H=1/2$), where the weight (which is the long memory kernel) expressed as a function have the same mean and covariance of FBM as following [6]:

$$\begin{gathered}\phi(s,t)=\phi_H(s,t)=H(2H-1)|s-t|^{2H-2} \quad , \quad s,t\in R \\ \int_0^t\int_0^s \phi(u,v)\,du\,dv=\frac{1}{2}\left(t^{2H}+s^{2H}-|s-t|^{2H}\right)=E\left(W_t^H W_s^H\right)\end{gathered} \tag{7}$$

The standard Brownian motion is a Markovian process and free from Arbitrage (the simultaneous buying and selling of securities, currency, or commodities in different markets or in derivative forms in order to take advantage of different prices for the same asset).

### 1.2. Multidimensional processes

An example of a system has adopted using three stochastic differential equations for the triple variables $\left(X_{1,t},X_{2,t},X_{3,t}\right)$ driven by three independent Brownian motions $\left(W_{1,t}^{H_1},W_{2,t}^{H_2},W_{3,t}^{H_3}\right)$[7].

$$\begin{aligned}dX_{1,t}&=-2X_{1,t}dt+dW_{1,t}^{H_1}+X_{2,t}dW_{2,t}^{H_2}+X_{3,t}dW_{3,t}^{H_3}\\ dX_{2,t}&=-\left(X_{1,t}+2X_{2,t}-X_{3,t}\right)dt+2X_{1,t}dW_{1,t}^{H_1}+3dW_{2,t}^{H_2}+dW_{3,t}^{H_3}\\ dX_{3,t}&=X_{3,t}dt+0.3X_{2,t}dW_{1,t}^{H_1}+dW_{2,t}^{H_2}\end{aligned}$$

This system can be organized to matrix form with a drift vector expressions and a diffusion matrix as follows:

$$\begin{pmatrix} dX_{1,t} \\ dX_{2,t} \\ dX_{3,t} \end{pmatrix} = \begin{pmatrix} -2X_{1,t} \\ -\left(X_{1,t} + 2X_{2,t} - X_{3,t}\right) \\ X_{3,t} \end{pmatrix} dt + \begin{pmatrix} 1 & X_{2,t} & X_{3,t} \\ 2X_{1,t} & 3 & 1 \\ 0.3X_{2,t} & 1 & 0 \end{pmatrix} \begin{pmatrix} dW_{1,t}^{H_1} \\ dW_{2,t}^{H_2} \\ dW_{3,t}^{H_3} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ -1 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} dt + \begin{pmatrix} 1 & X_{2,t} & X_{3,t} \\ 2X_{1,t} & 3 & 1 \\ 0.3X_{2,t} & 1 & 0 \end{pmatrix} \begin{pmatrix} dW_{1,t}^{H_1} \\ dW_{2,t}^{H_2} \\ dW_{3,t}^{H_3} \end{pmatrix}$$

This system can be solved analytically, if we can diagonalize the drift and diffusion matrices. This is impermissible, because there is no sharable null space for two different matrices. So, the maximum search method is the only way to solve this system of equations.

### 1.3. Maximum likelihood estimator MLE

The best estimator often can be obtained by MLE which have many desired properties, such as asymptotic unbiasedness, and asymptotic efficiency. This means it attains Cramer-Rao lower bound, and has asymptotic normal distribution. The maximum likelihood method depends on the Gaussian likelihood function assumption. This mean it considers only the mean and variance of estimators. The maximum likelihood can be expressed by the joint normal distribution, which is equivalent to the multivariate normal with mean vector $\underline{\mu}$ and variance-covariance matrix $\Omega$ .

Let a multivariate diffusion process consists of $p$ variable such that:

$$\begin{pmatrix} dx_{1,t} \\ \vdots \\ dx_{p,t} \end{pmatrix} = \begin{pmatrix} \mu_1\left(x_{1,t}, t\right) \\ \vdots \\ \mu_p\left(x_{p,t}, t\right) \end{pmatrix} dt + \Sigma^{\frac{1}{2}}\left(x_{i,t}, x_{j,t}, t\right) \begin{pmatrix} dW_{1,t}^{H_1} \\ \vdots \\ dW_{p,t}^{H_p} \end{pmatrix}$$

$$\begin{pmatrix} dW_{1,t}^{H_1} \\ \vdots \\ dW_{p,t}^{H_p} \end{pmatrix} = \Sigma^{-\frac{1}{2}}\left(x_{i,t}, x_{j,t}, t\right) \left( \begin{pmatrix} dx_{1,t} \\ \vdots \\ dx_{p,t} \end{pmatrix} - \begin{pmatrix} \mu_1\left(x_{1,t}, t\right) \\ \vdots \\ \mu_p\left(x_{p,t}, t\right) \end{pmatrix} dt \right)$$

and the joint distribution is:

$$p(\underline{x};\underline{\theta},\underline{H}) = \frac{1}{(2\pi)^{\frac{np}{2}} \left|\Sigma\left(\underline{x}_{i,t}, \underline{x}_{j,t}, t\right) \otimes \Omega_H\right|^{\frac{1}{2}}} \exp\left\{ \begin{array}{l} -\frac{1}{2} vec\left(d\underline{x}_1 - \mu_1(\underline{x}_1, t), \ldots, d\underline{x}_p - \mu_p(\underline{x}_p, t)\right)' \\ \left(\Sigma\left(\underline{x}_{i,t}, \underline{x}_{j,t}, t\right) \otimes \Omega_H\right)^{-1} vec\left(d\underline{x}_1 - \mu_1(\underline{x}_1, t), \ldots, d\underline{x}_p - \mu_p(\underline{x}_p, t)\right) \end{array} \right\}$$

Where $vec$ denotes vector with all observations of $p$ variables with sample size $n$ stacked vertically, $\otimes$ is matrix Kronecker product, $\Omega_H$ is variance covariance block matrix of FGN, $\Sigma\left(\underline{x}_{i,t}, \underline{x}_{j,t}, t\right)$ is the diffusion matrix, and $\underline{\theta}$ is a vector contains all the parameters of drift and diffusion functions.

## 2. Application

The maximum likelihood method is used to fit a system of three fractional stochastic differential equations, with daily three banking sector stock prices from January 1, 2010 to 11 March 11, 2019, as shown in Figure 1. The first step is to find the form of drift and diffusion functions using the Langevin method [8], by calculating the conditional moments and determining the drift and diffusion forms. We want here to refer to that we could not determine the conditional moment of three variables simultaneously, because the conditional transition density of three variables is so difficult to obtain. Accordingly, we use a pair of variables each time using the previously suggested method and numerical searching for the maximum likelihood to find optimal parameter estimates. The search takes long time because the high dimensionality of model parameters. The first and second conditional moments show a damped second order sine function and a damped second order polynomial respectively as depicted in Figures 3-7.

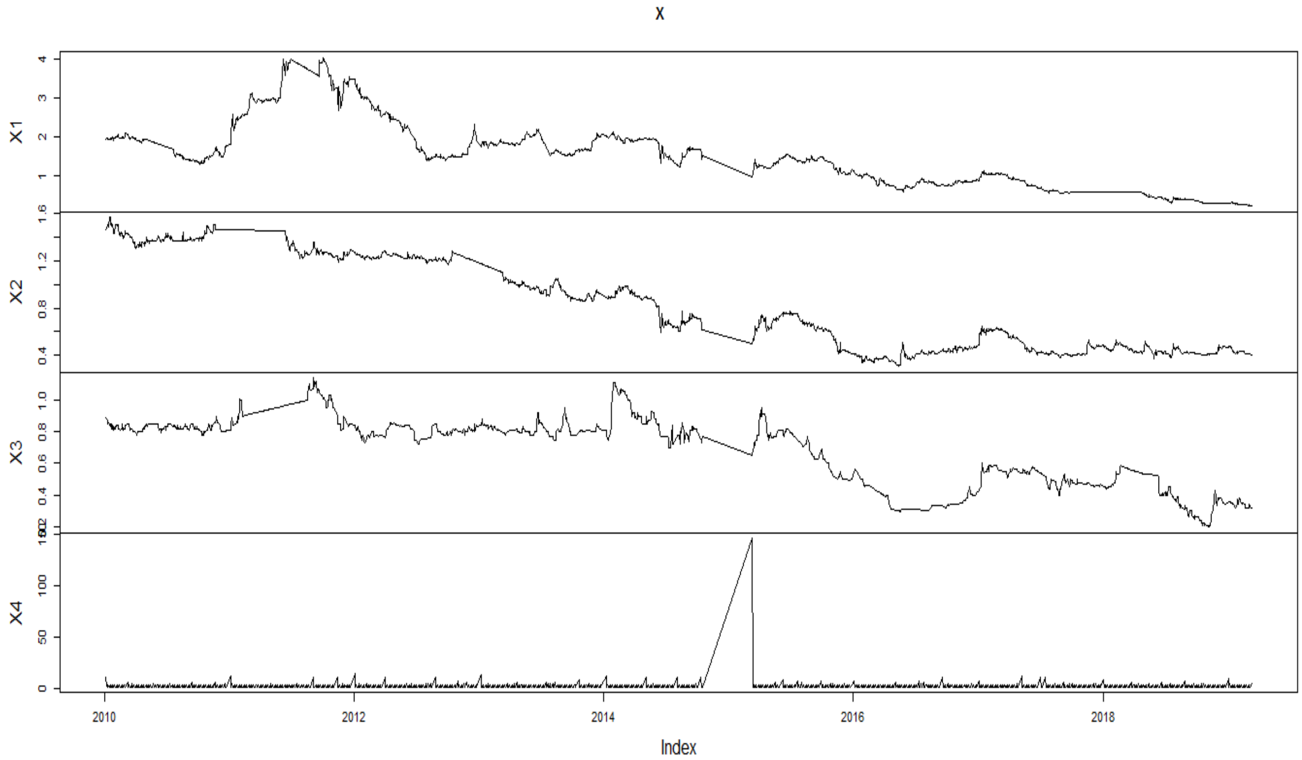


Figure 1. Plot of the stock prices time series and time between observations (fourth panel)

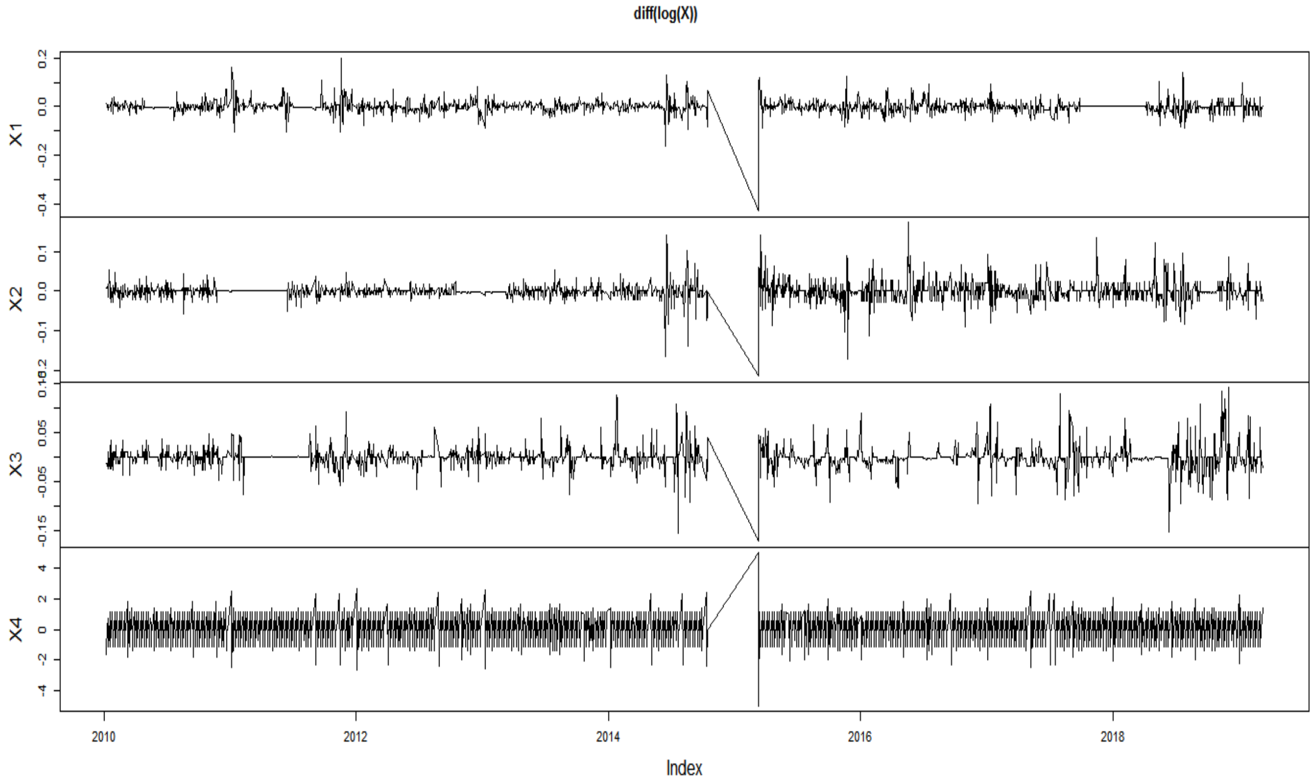


Figure 2. Plot of the returns of three series with time difference (fourth panel)

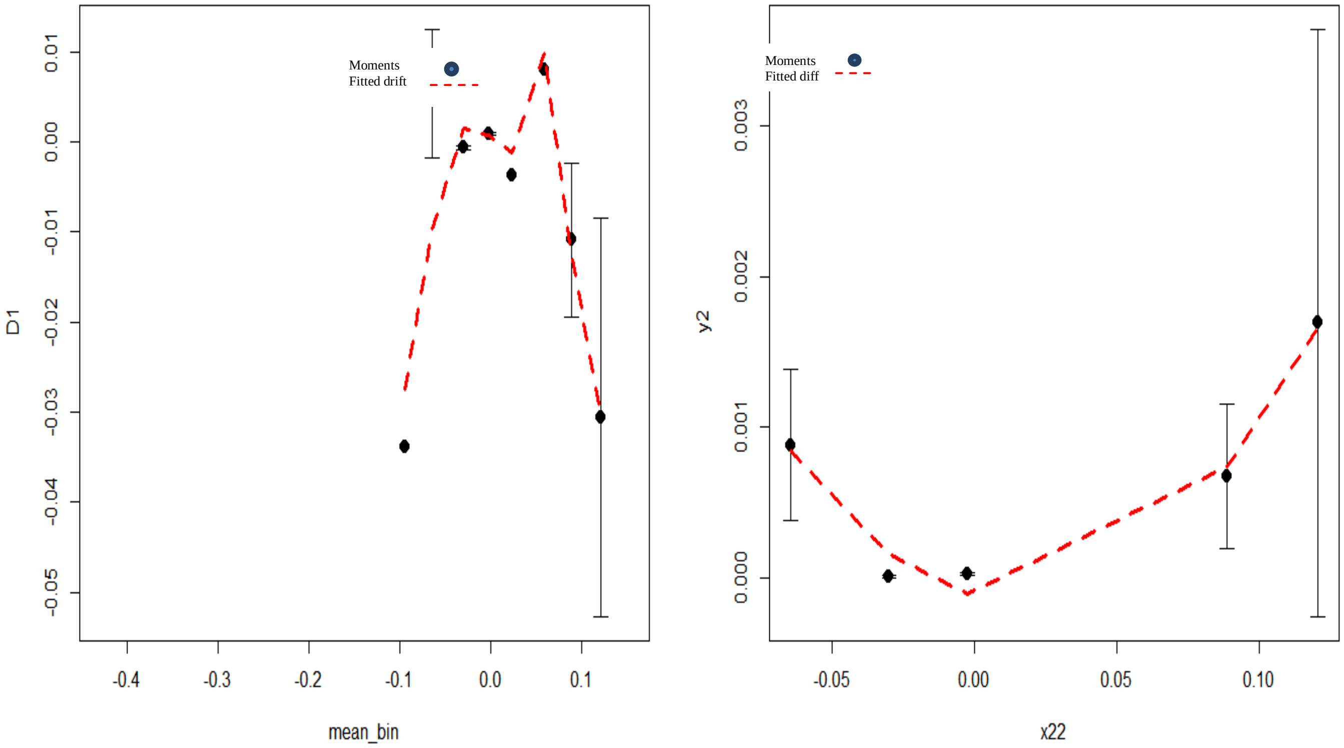


Figure 3. Drift and diffusion by Langevin method for series 1, conditional moments (black), error bar (bars), drift fitting (red)

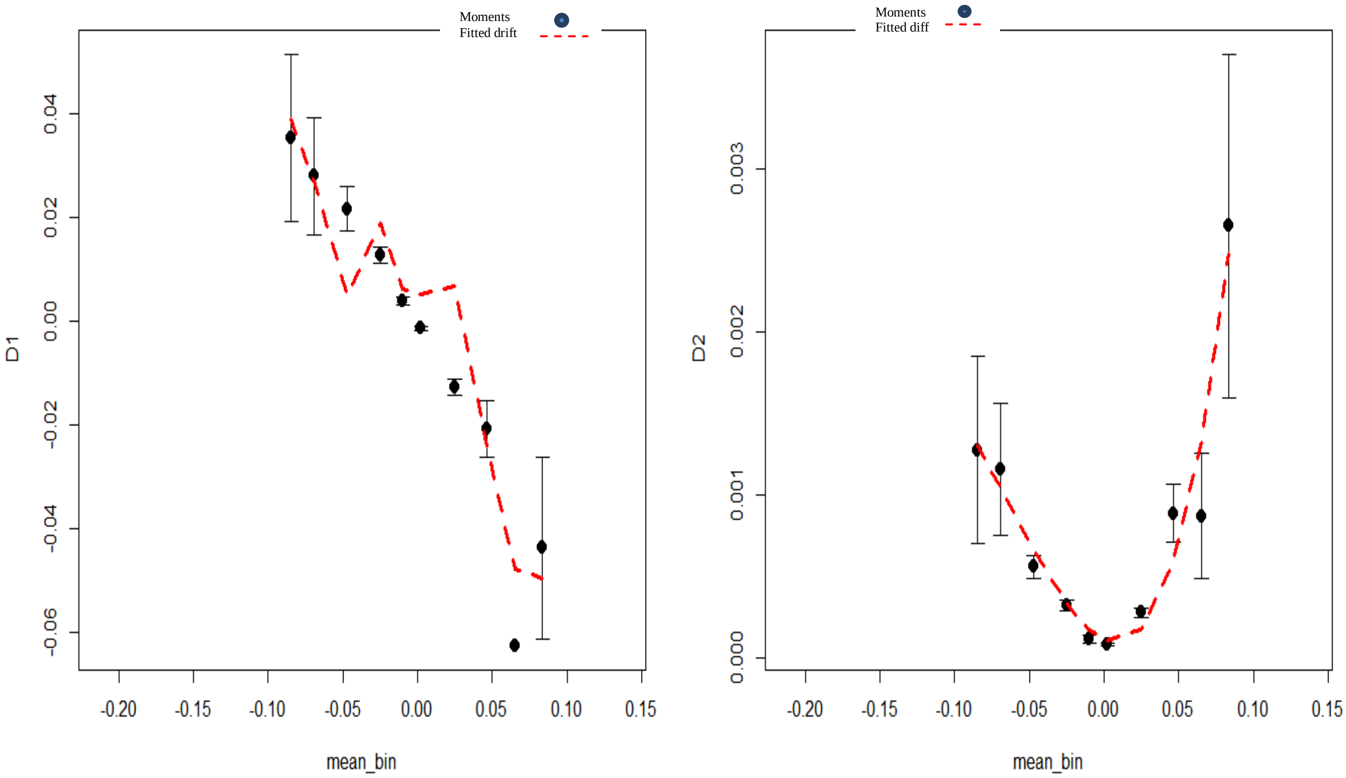


Figure 4. Drift and diffusion by Langevin method for series 2, conditional moments (black) ,error bar (bars), and drift fitting (red)

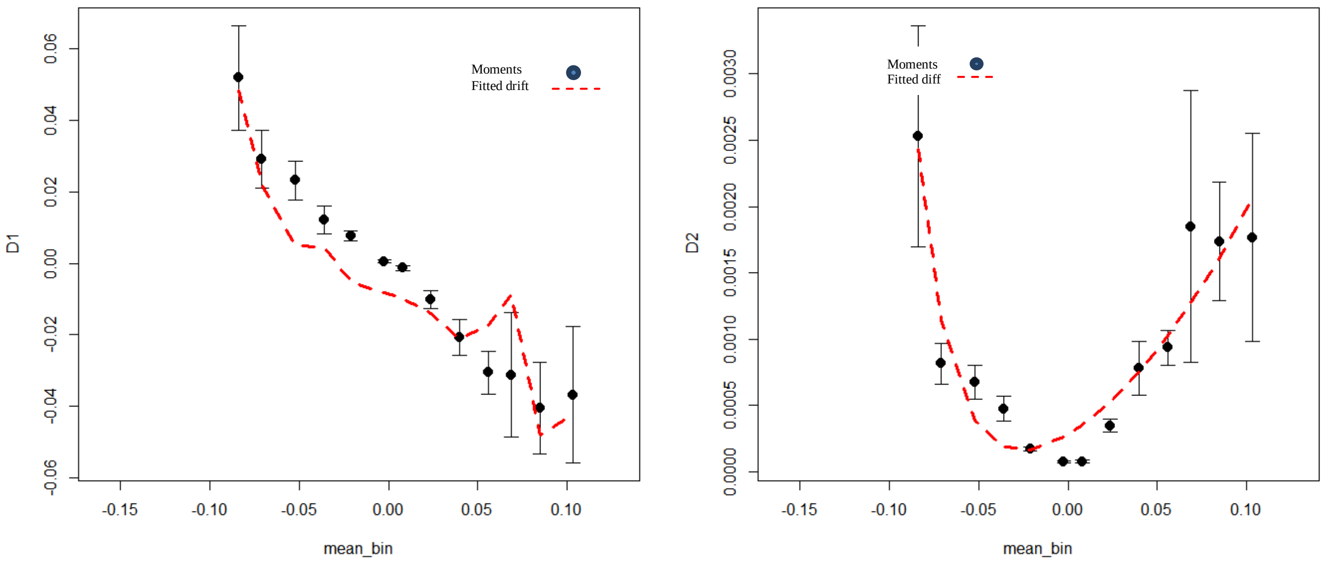


Figure 5. Drift and diffusion by Langevin method for series 3, conditional moments (black) ,error bar (bars), and drift fitting (red)

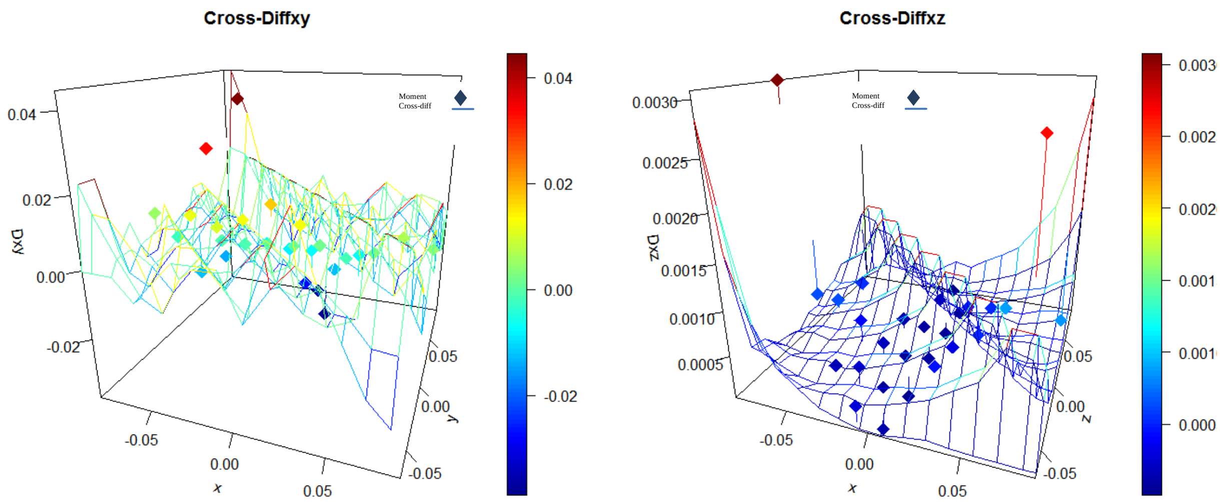


Figure 6. Cross-diffusion by Langevin method for series 1,2 and series 1,3, cross conditional moments (colored points), and diffusion fitting (colored surface)

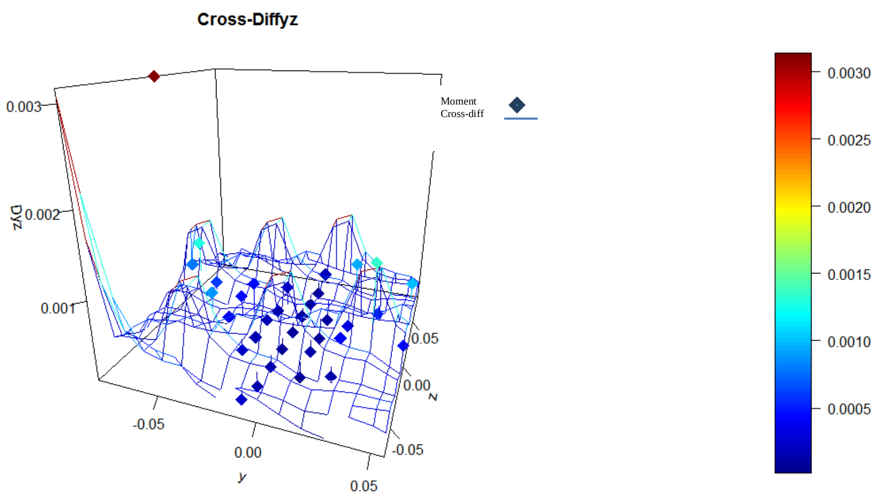


Figure 7. Cross-diffusion by Langevin method for series 2,3, cross conditional moments (colored points), and diffusion fitting (colored surface)

To decrease the search time, we use the nonlinear least squares in R program to determine the initial values for numerical search optimization method. We see from Figures 3-7, that the drifts and diffusions are not linear and they follow second order equation with a damping factor.

### 2.1. The models of FSDE

From the fitted conditional moments in Figures 3-7, we find the best functional forms of drift and diffusion for return rate $r_t = \log(x_t / x_{t-1})$ [9], as follow:

Drift function:

$$\mathrm{E}(dr_i) = f_i(\underline{r}, \theta)dt = \theta_{i0} r_i^2 \sin(\theta_{i1} r_i + \theta_{i2}) + \theta_{i3} \exp(\theta_{i4} r_i)dt \quad , \ i = 1,2,3$$

$$\mathrm{E}(r_{t+1} - r_t) = \theta_{i0} r_{ti}^2 \sin(\theta_{i1} r_{ti} + \theta_{i2}) + \theta_{i3} \exp(\theta_{i4} r_{ti})dt$$

$$\mathrm{E}\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\right) - \log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right) = \theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2 \sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \theta_{i2}\right) + \theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t$$

$$\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}} \Big/ \frac{x_{i,t}}{x_{i,t-1}}\right)\right) = \theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2 \sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \theta_{i2}\right) + \theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t$$

$$\mathrm{E}\left(\log\left(\frac{x_{i,t+1}x_{i,t-1}}{x_{i,t}^2}\right)\right) = \theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2 \sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \theta_{i2}\right) + \theta_{i3}\exp\left(\theta_{i4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\Delta t$$

$$\mathrm{E}(x_{i,t+1}) = \frac{x_{i,t}^2}{x_{i,t-1}} e^{\theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2 \sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \theta_{i2}\right) + \theta_{i3}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\theta_{i4}} \Delta t}$$

Where $\lim_{\Delta t \to 0} \Delta t = dt$

Diffusion function:

$$g_{ij}(\underline{r}, \alpha) = \sqrt{\left(\alpha_{ij0}(r_i + \alpha_{ij1})^2 + \alpha_{ij2}\exp(\alpha_{ij3} r_i)\right)} \qquad i = 1,2,3 \ , j = 1,2,3, i = j$$

$$\mathrm{E}\left((r_{t+1} - r_t)^2\right) = \left(\alpha_{ij0}(r_i + \alpha_{ij2})^2 + \alpha_{ij2}\exp(\alpha_{ij3} r_i)\right) E\left(W_{t+1}^{H_i} - W_t^{H_j}\right)^2$$

$$\mathrm{E}\left(\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\right) - \log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\right) = \left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \alpha_{ij1}\right)^2 + \alpha_{ij2}\exp\left(\alpha_{ij3}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\right)(\Delta t)^{2H_i}$$

$$\mathrm{E}\left(\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}} \Big/ \frac{x_{i,t}}{x_{i,t-1}}\right)\right)^2\right) = \left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \alpha_{ij1}\right)^2 + \alpha_{ij2}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\alpha_{ij3}}\right)(\Delta t)^{2H_i}$$

$$\mathrm{E}\left(\left(\log\left(\frac{x_{i,t+1}x_{i,t-1}}{x_{i,t}^2}\right)\right)^2\right) = \left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \alpha_{ij1}\right)^2 + \alpha_{ij2}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\alpha_{ij3}}\right)(\Delta t)^{2H_i}$$

$$\mathrm{E}(x_{i,t+1}^2) = \frac{x_{i,t}^4}{x_{i,t-1}^2} e^{\left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right) + \alpha_{ij1}\right)^2 + \alpha_{ij2}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\alpha_{ij3}}\right)(\Delta t)^{2H_i}}$$

$$\operatorname{var}\left(x_{i,t+1}\right)=\frac{x_{i,t}^{4}}{x_{i,t-1}^{2}}e^{\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij1}\right)^{2}+\alpha_{ij2}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\alpha_{ij3}}(\Delta t)^{2H_i}}-\left(E\left(x_{i,t+1}\right)\right)^{2}$$

$$=\frac{x_{i,t}^{4}}{x_{i,t-1}^{2}}e^{2\sqrt{\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij1}\right)^{2}+\alpha_{ij2}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\alpha_{ij3}}}(\Delta t)^{2H_i}}-\frac{x_{i,t}^{4}}{x_{i,t-1}^{2}}e^{2\theta_{i0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)^{2}\sin\left(\theta_{i1}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\theta_{i2}\right)+2\theta_{i3}\left(\frac{x_{i,t}}{x_{i,t-1}}\right)^{\theta_{i4}}\Delta t}$$

$$g_{ij}\left(\underline{r},\alpha\right)=\left(\alpha_{ij0}\left(r_i+\alpha_{ij1}\right)\left(r_j+\alpha_{ij2}\right)+\alpha_{ij3}\exp\left(\alpha_{ij4}r_i+\alpha_{ij5}r_j\right)\right)\left(W_{t+1}^{H_i}-W_t^{H_i}\right)\left(W_{t+1}^{H_j}-W_t^{H_j}\right)\quad i=1,2,3\ ,\ j=1,2,3, i\neq j$$

$$g_{ij}\left(\underline{dr_i dr_j},\alpha\right)=\left(\alpha_{ij0}\left(r_i+\alpha_{ij1}\right)\left(r_j+\alpha_{ij2}\right)+\alpha_{ij3}\exp\left(\alpha_{ij4}r_i+\alpha_{ij5}r_j\right)\right)\left(W_{t+1}^{H_i}-W_t^{H_i}\right)\left(W_{t+1}^{H_j}-W_t^{H_j}\right)$$

$$\mathrm{E}\left(\left(r_{i,t+1}-r_{i,t}\right)\left(r_{i,t+1}-r_{i,t}\right)\right)=\left(\alpha_{ij0}\left(r_i+\alpha_{ij1}\right)\left(r_j+\alpha_{ij2}\right)+\alpha_{ij3}\exp\left(\alpha_{ij4}r_i+\alpha_{ij5}r_j\right)\right)E\left(\left(W_{t+1}^{H_i}-W_t^{H_i}\right)\left(W_{t+1}^{H_j}-W_t^{H_j}\right)\right)$$

$$\mathrm{E}\left(\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\right)-\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\right)-\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)\right)\right)=$$

$$\left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij1}\right)\left(\log\left(\frac{x_{j,t}}{x_{j,t-1}}\right)+\alpha_{ij2}\right)+\alpha_{ij3}\exp\left(\alpha_{ij4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij5}\log\left(\frac{x_{j,t}}{x_{j,t-1}}\right)\right)\right)\left(\frac{1}{2}\left[\left(k_1+1\right)^{2H_{ij}}-\left(k_1+k_2\right)^{2H_{ij}}+\left(k_2-1\right)^{2H_{ij}}\right]\right)$$

$$\mathrm{E}\left(\log\left(\frac{x_{i,t+1}}{x_{i,t}}\Big/\frac{x_{i,t}}{x_{i,t-1}}\right)\log\left(\frac{x_{j,t+1}}{x_{j,t}}\Big/\frac{x_{j,t}}{x_{j,t-1}}\right)\right)=$$

$$\left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij1}\right)\left(\log\left(\frac{x_{j,t}}{x_{j,t-1}}\right)+\alpha_{ij2}\right)+\alpha_{ij3}\exp\left(\alpha_{ij4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij5}\log\left(\frac{x_{j,t}}{x_{j,t-1}}\right)\right)\right)\left(\frac{1}{2}\left[\left(k_1+1\right)^{2H_{ij}}-\left(k_1+k_2\right)^{2H_{ij}}+\left(k_2-1\right)^{2H_{ij}}\right]\right)$$

$$\mathrm{E}\left(\log\left(\frac{x_{i,t+1}x_{i,t-1}}{x_{i,t}^{2}}\right)\log\left(\frac{x_{j,t+1}x_{j,t-1}}{x_{j,t}^{2}}\right)\right)=$$

$$\left(\alpha_{ij0}\left(\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij1}\right)\left(\log\left(\frac{x_{j,t}}{x_{j,t-1}}\right)+\alpha_{ij2}\right)+\alpha_{ij3}\exp\left(\alpha_{ij4}\log\left(\frac{x_{i,t}}{x_{i,t-1}}\right)+\alpha_{ij5}\log\left(\frac{x_{j,t}}{x_{j,t-1}}\right)\right)\right)\left(\frac{1}{2}\left[\left(k_1+1\right)^{2H_{ij}}-\left(k_1+k_2\right)^{2H_{ij}}+\left(k_2-1\right)^{2H_{ij}}\right]\right)$$

Where $k_1$ and $k_2$ are times corresponding to $x_{i,t+1}$ and $x_{j,t}$, respectively .

We use the discrete form of model to fit the data, and it could be used for forecasting or for portfolio building (investment strategy). Additionally, we can solve the model analytically and then estimate the parameter. However, this is not an easy task as it is frequently in non-closed form.
The next step is to estimate the parameters of the model that have maximum log likelihood.

## 3. Numerical calculation

We analyzed three stock prices time series. The data is of size 2030, taken from 1 January 2010 to 11 March 2019. Since the data is positive prices, we take logarithm difference $r\left(t\right)=\log\left(p\left(x_{t+1}\right)/p\left(x_t\right)\right)$ to transform it to return rate, which is very important in finance investment, and to make series approximately normal distribution. The time difference is $\Delta t=1$, which reflects the difference of working days in year. We fit the return with appropriate models for drift and diffusion to extract a primary models for fractional stochastic differential equation by plotting the scatter of the first difference of return with lagged return, and estimating the parameter of the model using the nonlinear least square. This will give us a glimpse about model function as shown below [9]:

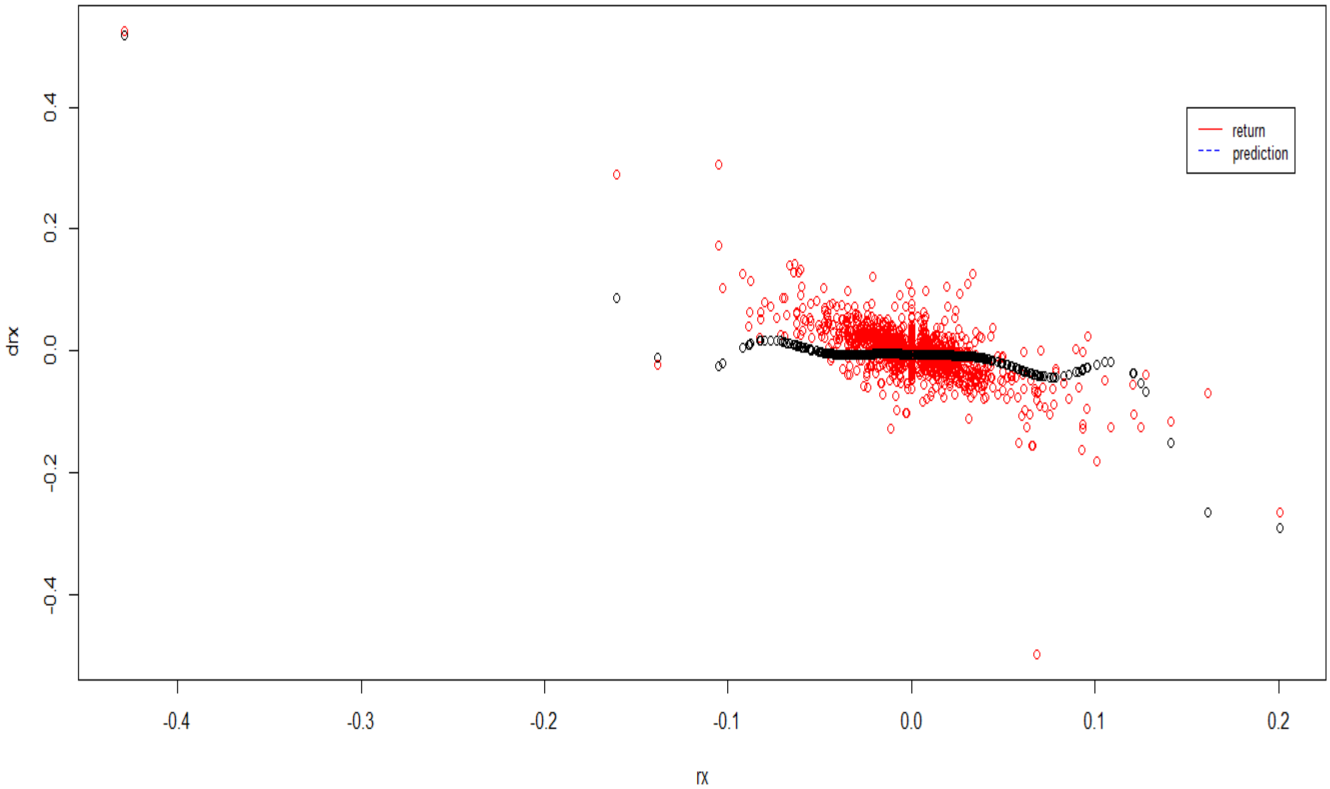


Figure 8. Return scatter plot with fitted drift model for series 1, return (red), and fitted drift(black)

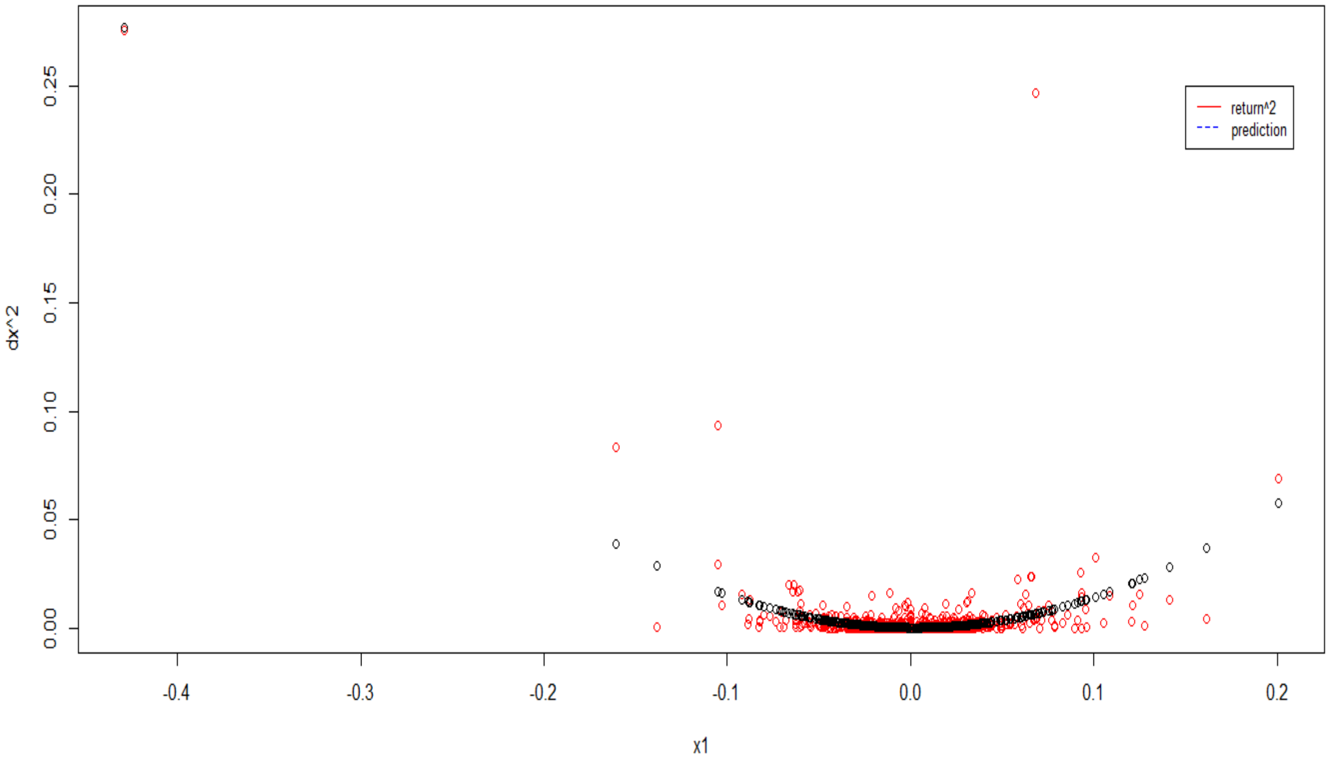


Figure 9. Return scatter plot with fitted diffusion model for series 1, square return (red), and fitted diffusion (black)

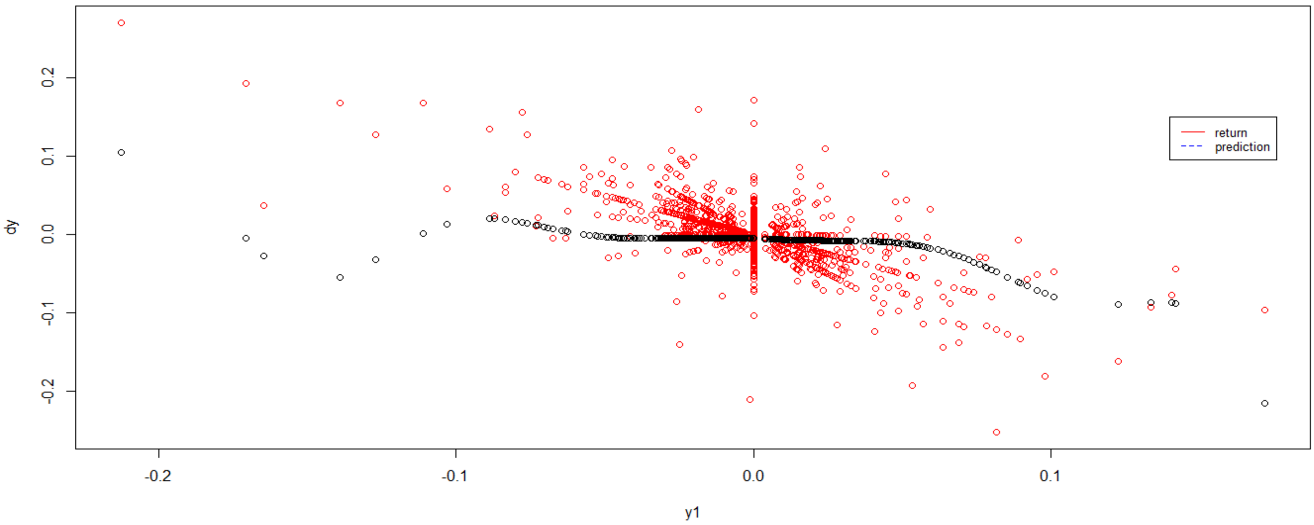


Figure 10. Return scatter plot with fitted drift model for series 2 return (red), and fitted drift (black)

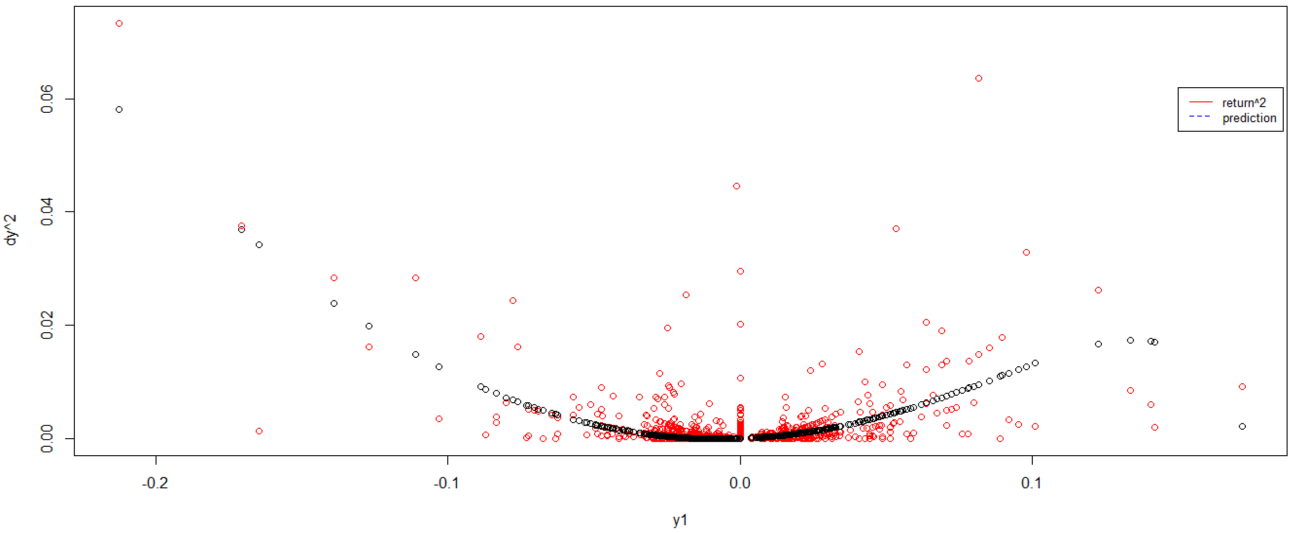


Figure 11. Return scatter plot with fitted diffusion model for series 2, square return (red), and fitted diffusion (black)

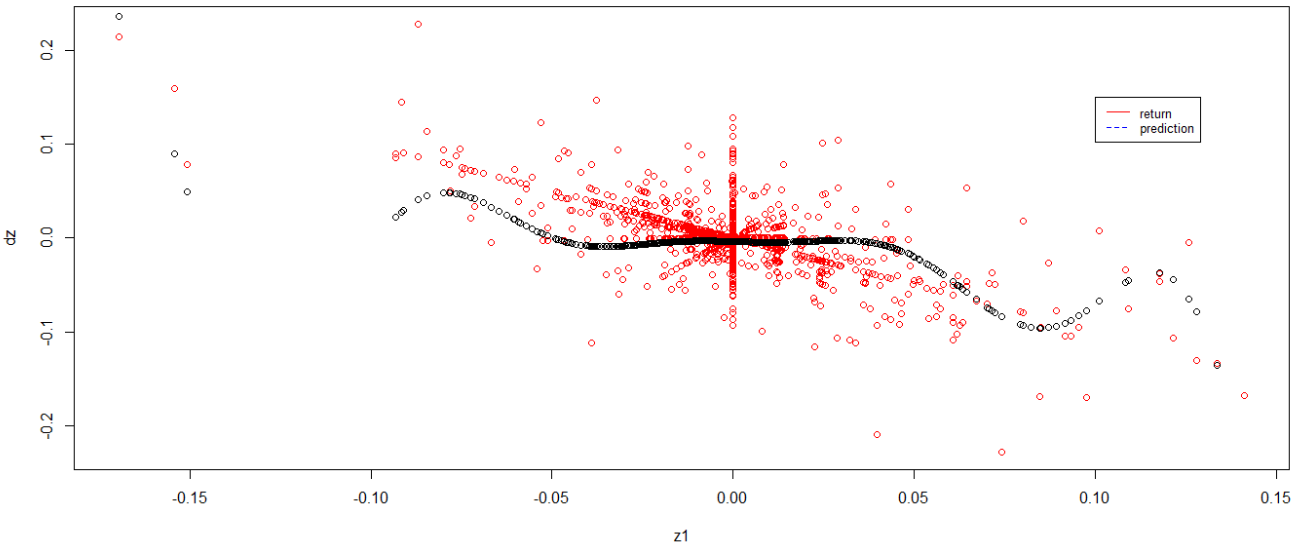


Figure 12. Return scatter plot with fitted drift model for series 3, return (red), fitted drift (black)

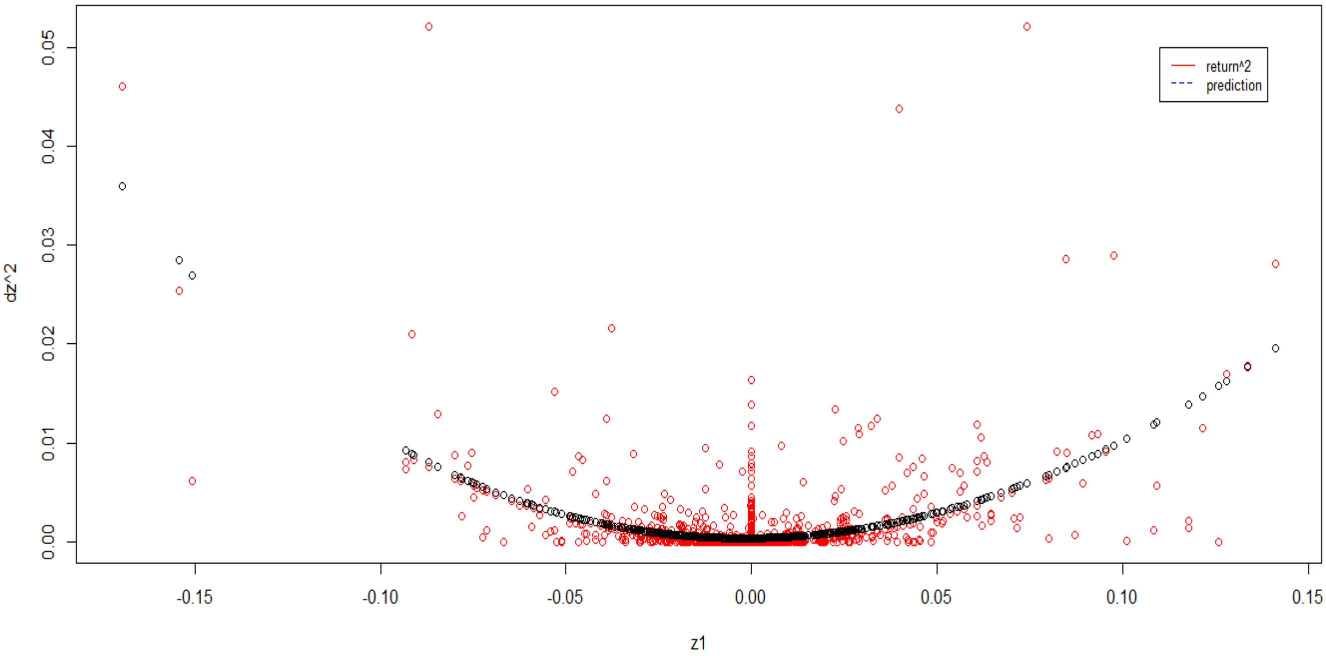


Figure 13. Return scatter plot with fitted diffusion model for series 3, square return (red), and fitted diffusion (black)

From Figures 8 -13, we perceive that the models represent the data in an accurate manner, and we can use them in the model. Secondly, we have estimated the parameters with Hurst indices simultaneously to obtain the final fitted model as follow:

$$\begin{pmatrix} X_{1,t+1} \\ X_{2,t+1} \\ X_{3,t+1} \end{pmatrix} = \begin{pmatrix} \dfrac{x_{1,t}^{2}}{x_{1,t-1}} e^{\theta_{10}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)\right)^2 \sin\left(\theta_{11}\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\theta_{12}\right)+\theta_{13}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\theta_{14}}\Delta t+\left(\alpha_{110}\left(\log\left(\frac{x_{1,t}}{x_{1,t-1}}\right)+\alpha_{111}\right)^2+\alpha_{112}\left(\frac{x_{1,t}}{x_{1,t-1}}\right)^{\alpha_{113}}\right)\left(W_{t+1}^{H_1}-W_t^{H_1}\right)} \\ \dfrac{x_{2,t}^{2}}{x_{2,t-1}} e^{\theta_{20}\left(\log\left(\frac{x_{2,t}}{x_{2,t-1}}\right)\right)^2 \sin\left(\theta_{21}\log\left(\frac{x_{2,t}}{x_{2,t-1}}\right)+\theta_{22}\right)+\theta_{23}\left(\frac{x_{2,t}}{x_{2,t-1}}\right)^{\theta_{24}}\Delta t+\left(\alpha_{220}\left(\log\left(\frac{x_{2,t}}{x_{2,t-1}}\right)+\alpha_{221}\right)^2+\alpha_{222}\left(\frac{x_{2,t}}{x_{2,t-1}}\right)^{\alpha_{223}}\right)\left(W_{t+1}^{H_2}-W_t^{H_2}\right)} \\ \dfrac{x_{3,t}^{2}}{x_{3,t-1}} e^{\theta_{30}\left(\log\left(\frac{x_{3,t}}{x_{3,t-1}}\right)\right)^2 \sin\left(\theta_{31}\log\left(\frac{x_{3,t}}{x_{3,t-1}}\right)+\theta_{32}\right)+\theta_{33}\left(\frac{x_{3,t}}{x_{3,t-1}}\right)^{\theta_{34}}\Delta t+\left(\alpha_{330}\left(\log\left(\frac{x_{3,t}}{x_{3,t-1}}\right)+\alpha_{331}\right)^2+\alpha_{332}\left(\frac{x_{3,t}}{x_{3,t-1}}\right)^{\alpha_{333}}\right)\left(W_{t+1}^{H_3}-W_t^{H_3}\right)} \end{pmatrix}$$

Where $\left(W_{t+1}^{H_i}-W_t^{H_i}\right) \sim N\left(0,\Omega_{H_i}\right)$ .

Table 1 shows the parameter estimation of drift and diffusion function with Hurst indices, and mean square error (MSE)

Table 1. Parameter estimation

| equations | $\theta_{0i}$ | $\theta_{1i}$ | $\theta_{2i}$ | $\theta_{3i}$ | $\theta_{4i}$ |
|---|---|---|---|---|---|
| Drift1 | 8.483829 | 193.2458 | -17.27928 | 0.002263530 | -20.20949 |
| Drift2 | 14.15109 | 189.1261 | -33.04932 | -0.002592704 | 43.64039 |
| Drift3 | -7.298171 | 36.60272 | 26.96703 | 0.001724703 | -32.86951 |
| | $\alpha_{0i}$ | $\alpha_{1i}$ | $\alpha_{2i}$ | $\alpha_{3i}$ | |
| Dif1 | 59.42193 | -0.001110057 | 0.000307388 | -7.896018 | |

| equations | $\theta_{0i}$ | $\theta_{1i}$ | $\theta_{2i}$ | $\theta_{3i}$ | $\theta_{4i}$ |
|---|---|---|---|---|---|
| Dif2 | 0.002884926 | 0.008110351 | 0.008687212 | -0.000000018 | |
| Dif3 | 24.47965 | 39.57649 | -0.00006765 | -0.00001165 | |

Table 2. Parameter estimation

| | $\alpha_{0ij}$ | $\alpha_{1ij}$ | $\alpha_{2ij}$ | $\alpha_{3ij}$ | $\alpha_{4ij}$ | $\alpha_{5ij}$ |
|---|---|---|---|---|---|---|
| Dif12 | -0.0003047 | 0.0000003 | -0.03047014 | -0.00345 | 9.391377 | 0.20524 |
| Dif13 | 0.01152 | 9.303704 | 0.0001568 | 0.01361126 | 0.06636 | 0.00009 |
| Dif23 | -4.52658 | -1.501161 | -0.0003446 | -0.1749375 | 0.0107 | -10.292 |
| | $H_1$ | $H_2$ | $H_3$ | $H_{12}$ | $H_{13}$ | $H_{23}$ |
| Hurst index | 0.7065334 | 0.7295484 | 0.8936673 | 0.6978811 | 0.436494 | 0.5824 |
| | Series1 | Series2 | Series3 | | | |
| MSE | 0.006041 | 0.00082889 | 0.000394256 | | | |

MSE (mean square error)

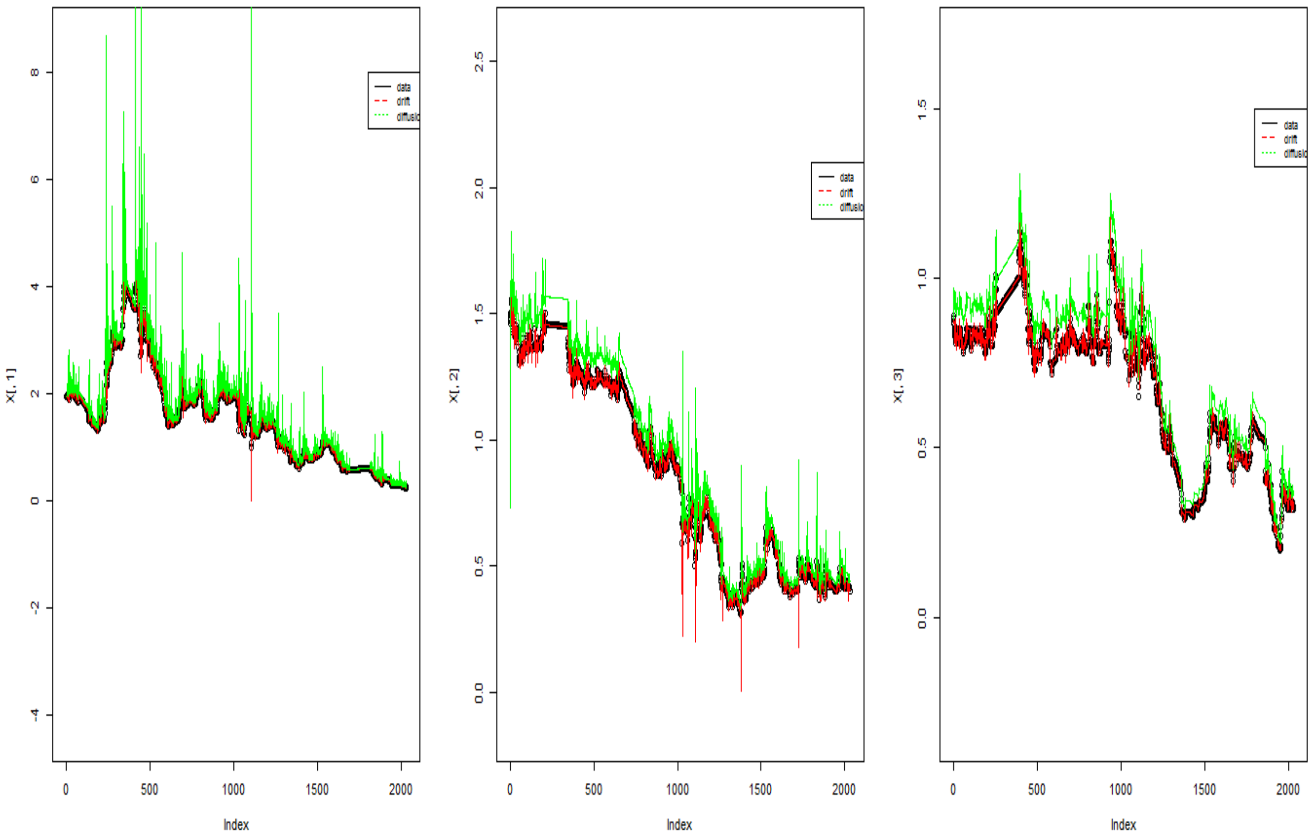


Figure 14. Plot of fitted models for 3 series (respectively), data (black points), fitted drift (red line), and fitted diffusion (green line)

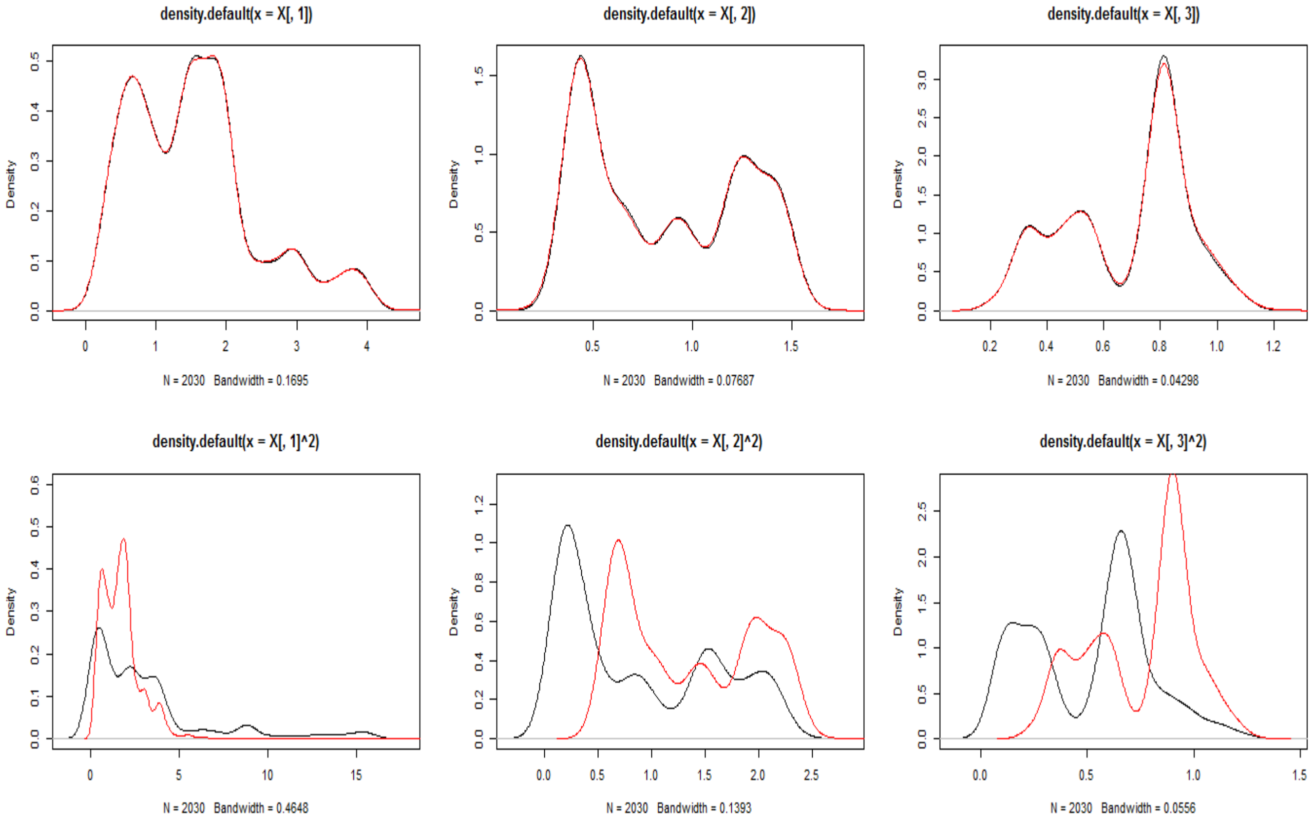


Figure 15. Plot of drift and diffusion densities with original data densities, data density (black line), and estimated density (red line)

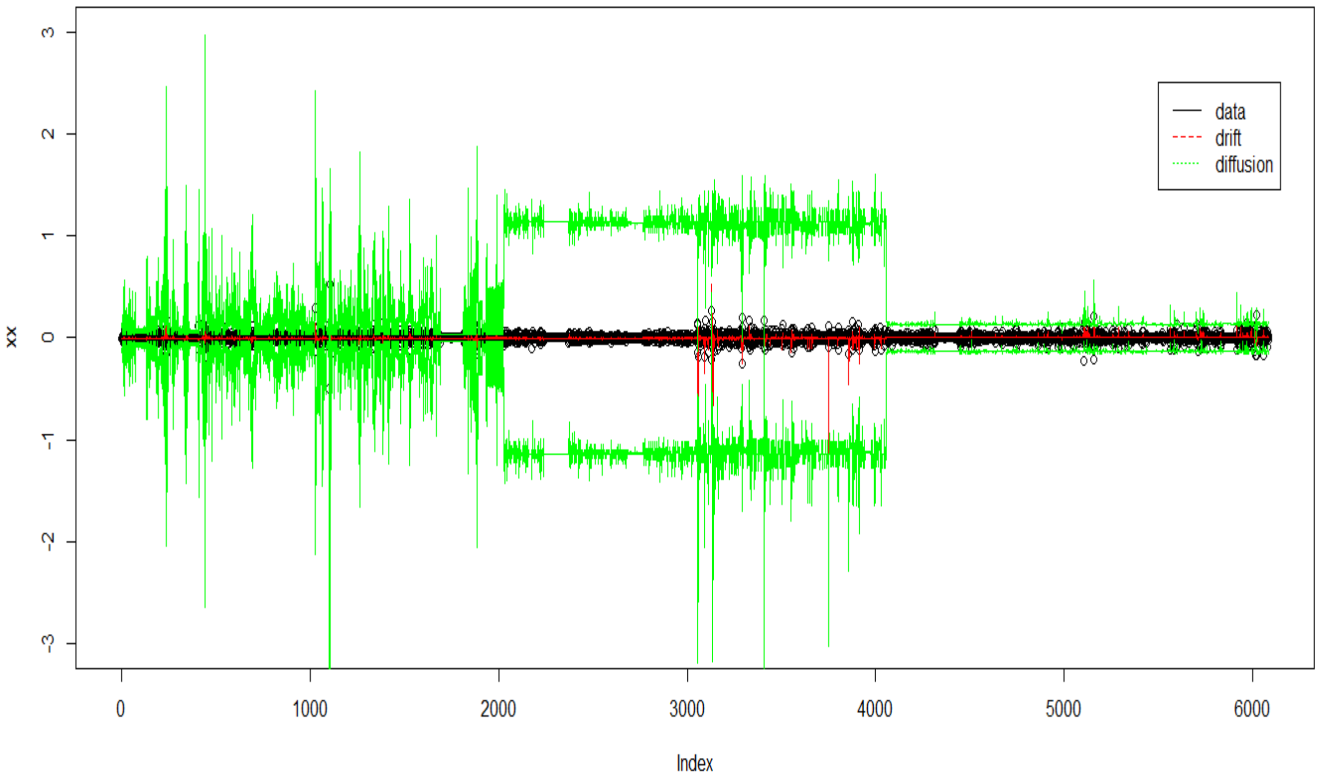


Figure 16. Plot of drift and diffusion 0.95% confidence interval of returns rate, returns rate (black points), fitted drift (red line), and 0.95% confidence interval (green line)

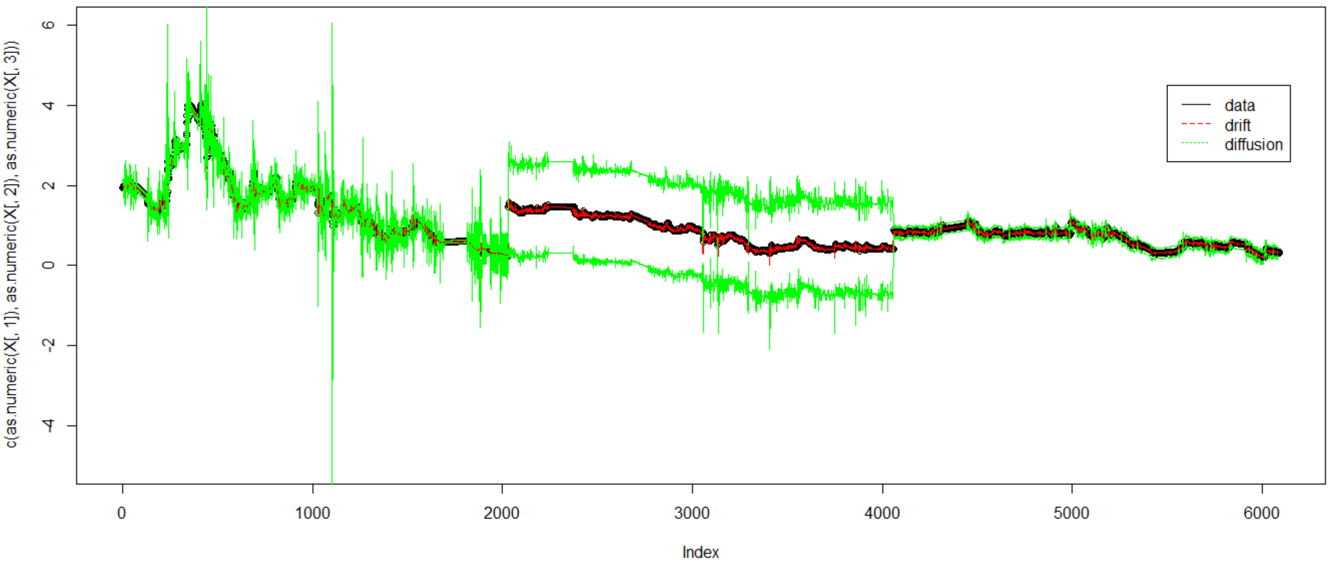


Figure 17. Plot of drift and diffusion 0.95% confidence interval of prices, prices (black points), fitted drift (red line), and 0.95% confidence interval (green line)

## 4. Conclusion

Fractional stochastic differential equations are a very efficient tool to represent the financial phenomena, because it reflects the dynamical behavior with long memory that is intrinsic characteristic of them. In this paper, we show the fitting dynamical model that superimposed other model in capturing the minute details in the data. The drift and diffusion are very important quantities in many applications especially in the financial portfolio building. The model specified shows many features in the data to be used to predict the future values and so build the portfolio. That is much benefited for the investor to overcome the risk of stock prices and to achieve a profit. The parameter of the model is estimated numerically by optimization method to maximize the logarithm of the likelihood using R program. The results in Table 1 show that Hurst indices are higher than 0.5 except $H_{13}$, and this mean there is a strong long memory behavior in three series. We see that from Figure 17, that the diffusion is very high (green line) in series 1, because of Hurst index is 0.7065334, and diffusion in series 3 is low because of high Hurst index. This reflects the long memory existence that will decrease the uncertainty and the prediction will be more accurate. In addition, the cross Hurst indices reflect the cross long memory correlation between different variables. As we conclude from Table 1, there is a cross long memory between series 1 and 2 and series 2 and 3, but there is negative short memory between series 2 and 3. Also, Figure 15 shows how the drift density is exactly fit the data density (first row), but the diffusion density (second row) has some dissimilarities, and the diffusion functions are not fit the data variance exactly.

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## Appendix

To construct the matrix of diffusion with the autocorrelation matrix of fractional Brownian motion, the vectorization of variables must include Kronecker product of variance-covariance matrix as in SURE (Seemingly Unrelated Regression Equations) model as below:

$$\Sigma \otimes I_n$$

$$if$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 \end{bmatrix}$$

$$vec(\Sigma * N(0,1)) = N(0, \Sigma \otimes I_3)$$

$$\Sigma \otimes I_3 = \begin{bmatrix} \sigma_1^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{12} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{13} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \sigma_{21} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_2^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{23} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \sigma_{31} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_{32} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \sigma_3^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_1^2 & 0 \\ 0 & 0 & \sigma_1^2 \end{bmatrix} & \begin{bmatrix} \sigma_{12} & 0 & 0 \\ 0 & \sigma_{12} & 0 \\ 0 & 0 & \sigma_{12} \end{bmatrix} & \begin{bmatrix} \sigma_{13} & 0 & 0 \\ 0 & \sigma_{13} & 0 \\ 0 & 0 & \sigma_{13} \end{bmatrix} \\ \begin{bmatrix} \sigma_{21} & 0 & 0 \\ 0 & \sigma_{21} & 0 \\ 0 & 0 & \sigma_{21} \end{bmatrix} & \begin{bmatrix} \sigma_2^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_2^2 \end{bmatrix} & \begin{bmatrix} \sigma_{23} & 0 & 0 \\ 0 & \sigma_{23} & 0 \\ 0 & 0 & \sigma_{23} \end{bmatrix} \\ \begin{bmatrix} \sigma_{31} & 0 & 0 \\ 0 & \sigma_{31} & 0 \\ 0 & 0 & \sigma_{31} \end{bmatrix} & \begin{bmatrix} \sigma_{32} & 0 & 0 \\ 0 & \sigma_{32} & 0 \\ 0 & 0 & \sigma_{32} \end{bmatrix} & \begin{bmatrix} \sigma_3^2 & 0 & 0 \\ 0 & \sigma_3^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix} \end{bmatrix}$$

$$
=\begin{bmatrix}
\begin{bmatrix}\sigma_1^2&0&0\\0&\sigma_1^2&0\\0&0&\sigma_1^2\end{bmatrix} & \begin{bmatrix}\sigma_1\rho_{12}\sigma_2&0&0\\0&\sigma_1\rho_{12}\sigma_2&0\\0&0&\sigma_1\rho_{12}\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_1\rho_{13}\sigma_3&0&0\\0&\sigma_1\rho_{13}\sigma_3&0\\0&0&\sigma_1\rho_{13}\sigma_3\end{bmatrix}\\
\begin{bmatrix}\sigma_1\rho_{12}\sigma_2&0&0\\0&\sigma_1\rho_{12}\sigma_2&0\\0&0&\sigma_1\rho_{12}\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_2^2&0&0\\0&\sigma_2^2&0\\0&0&\sigma_2^2\end{bmatrix} & \begin{bmatrix}\sigma_2\rho_{23}\sigma_3&0&0\\0&\sigma_2\rho_{23}\sigma_3&0\\0&0&\sigma_2\rho_{23}\sigma_3\end{bmatrix}\\
\begin{bmatrix}\sigma_1\rho_{13}\sigma_3&0&0\\0&\sigma_1\rho_{13}\sigma_3&0\\0&0&\sigma_1\rho_{13}\sigma_3\end{bmatrix} & \begin{bmatrix}\sigma_2\rho_{23}\sigma_3&0&0\\0&\sigma_2\rho_{23}\sigma_3&0\\0&0&\sigma_2\rho_{23}\sigma_3\end{bmatrix} & \begin{bmatrix}\sigma_3^2&0&0\\0&\sigma_3^2&0\\0&0&\sigma_3^2\end{bmatrix}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix} & \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{12}&0&0\\0&\rho_{12}&0\\0&0&\rho_{12}\end{bmatrix}\begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{13}&0&0\\0&\rho_{13}&0\\0&0&\rho_{13}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix}\\
\begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{12}&0&0\\0&\rho_{12}&0\\0&0&\rho_{12}\end{bmatrix}\begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix} & \begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix}\begin{bmatrix}\rho_{23}&0&0\\0&\rho_{23}&0\\0&0&\rho_{23}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix}\\
\begin{bmatrix}\sigma_1&0&0\\0&\sigma_1&0\\0&0&\sigma_1\end{bmatrix}\begin{bmatrix}\rho_{13}&0&0\\0&\rho_{13}&0\\0&0&\rho_{13}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} & \begin{bmatrix}\sigma_2&0&0\\0&\sigma_2&0\\0&0&\sigma_2\end{bmatrix}\begin{bmatrix}\rho_{23}&0&0\\0&\rho_{23}&0\\0&0&\rho_{23}\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix} & \begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\sigma_3&0&0\\0&\sigma_3&0\\0&0&\sigma_3\end{bmatrix}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\Sigma_{11}^{1/2}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2}\\
\Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2}\\
\Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\Sigma_{33}^{1/2}
\end{bmatrix}
$$

where $\Gamma_{ij}=\begin{bmatrix}\rho_{ij}&0&0\\0&\rho_{ij}&0\\0&0&\rho_{ij}\end{bmatrix}$

Here, $I_n$ matrix represents the autocorrelation matrix with $\phi=0$. So, if $\phi\neq 0$, we will have:

$$
\Omega=\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix}
$$

$$
vec(\Sigma * N\,(0,\Omega)) = N\,(0,\Sigma\otimes\Omega)
$$

$$
\Sigma\otimes\Omega=\begin{bmatrix}
\sigma_1^2\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{12}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{13}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix}\\
\sigma_{21}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_2^2\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{23}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix}\\
\sigma_{31}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_{32}\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix} & \sigma_3^2\begin{bmatrix}1&\phi&\phi^2\\\phi&1&\phi\\\phi^2&\phi&1\end{bmatrix}
\end{bmatrix}
$$

Now, if we suppose that every observation generated from different mean and variance, we have

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix} \quad \text{where } \Sigma \text{ is positive definite}$$

$$= \begin{bmatrix} \Sigma_{11} & \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{12}\Sigma_{22}^{1/2} & \Sigma_{22} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{13}\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}\Sigma_{33}^{1/2} & \Sigma_{33} \end{bmatrix}$$

$$vec(\Sigma * N(0,\Omega)) = N(0, \Sigma \otimes \Omega)$$

$$\Sigma \otimes \Omega = \begin{bmatrix} \Sigma_{11}^{1/2}\begin{bmatrix} 1 & \phi & \phi^2 \\ \phi & 1 & \phi \\ \phi^2 & \phi & 1 \end{bmatrix}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\begin{bmatrix} \rho & \phi & \phi^2 \\ \phi & \rho & \phi \\ \phi^2 & \phi & \rho \end{bmatrix}\Sigma_{22}^{1/2} & \Sigma_{12}^{1/2}\begin{bmatrix} \rho & \phi & \phi^2 \\ \phi & \rho & \phi \\ \phi^2 & \phi & \rho \end{bmatrix}\Sigma_{33}^{1/2} \\ \Sigma_{22}^{1/2}\begin{bmatrix} \rho & \phi & \phi^2 \\ \phi & \rho & \phi \\ \phi^2 & \phi & \rho \end{bmatrix}\Sigma_{11}^{1/2} & \Sigma_{22}^{1/2}\begin{bmatrix} 1 & \phi & \phi^2 \\ \phi & 1 & \phi \\ \phi^2 & \phi & 1 \end{bmatrix}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\begin{bmatrix} \rho & \phi & \phi^2 \\ \phi & \rho & \phi \\ \phi^2 & \phi & \rho \end{bmatrix}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\begin{bmatrix} \rho & \phi & \phi^2 \\ \phi & \rho & \phi \\ \phi^2 & \phi & \rho \end{bmatrix}\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\begin{bmatrix} \rho & \phi & \phi^2 \\ \phi & \rho & \phi \\ \phi^2 & \phi & \rho \end{bmatrix}\Sigma_{22}^{1/2} & \Sigma_{33}^{1/2}\begin{bmatrix} 1 & \phi & \phi^2 \\ \phi & 1 & \phi \\ \phi^2 & \phi & 1 \end{bmatrix}\Sigma_{33}^{1/2} \end{bmatrix}$$

Where $\phi$ is the autocorrelation and $\rho$ is the correlation between two different variables

$$\Sigma \otimes \Omega = \begin{bmatrix} \Sigma_{11}^{1/2}\Omega\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{12}.\,\Omega\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}.\,\Omega\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{12}.\,\Omega\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Omega\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}.\,\Omega\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{13}.\,\Omega\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}.\,\Omega\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\Omega\Sigma_{33}^{1/2} \end{bmatrix}$$

But the different variables cannot have the same autocorrelation, so the matrix will become:

$$\Sigma \otimes \Omega = \begin{bmatrix} \Sigma_{11}^{1/2}\Omega_{11}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{12}.\,\Omega_{12}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}.\,\Omega_{13}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{12}.\,\Omega_{21}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\,\Omega_{22}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}.\,\Omega_{23}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{13}.\,\Omega_{31}\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}.\,\Omega_{32}\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\,\Omega_{33}\Sigma_{33}^{1/2} \end{bmatrix}.$$ [1]

Now, we substitute $\phi$ with long memory dependence characterized by Hurst index. We get the following"

$$\Sigma \otimes \Omega_H = \begin{bmatrix} \Sigma_{11}^{1/2}\Omega_{H_1}\Sigma_{11}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{12}.\,\Omega_{H_{12}}\Sigma_{22}^{1/2} & \Sigma_{11}^{1/2}\Gamma_{13}.\,\Omega_{H_{13}}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{12}.\,\Omega_{H_{12}}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\,\Omega_{H_2}\Sigma_{22}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}.\,\Omega_{H_{23}}\Sigma_{33}^{1/2} \\ \Sigma_{11}^{1/2}\Gamma_{13}.\,\Omega_{H_{13}}\Sigma_{33}^{1/2} & \Sigma_{22}^{1/2}\Gamma_{23}.\,\Omega_{H_{23}}\Sigma_{33}^{1/2} & \Sigma_{33}^{1/2}\,\Omega_{H_3}\Sigma_{33}^{1/2} \end{bmatrix}$$

[1] Where $\Gamma$ represents a diagonal matrix of correlation coefficients and $\Gamma.\,\Omega$ represents an element by element multiplication of two matrices